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HADAMARD'S THEOREM FOR DIRICHLET SERIES

BY S. BOCHNER

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If two functions with the expansions

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$$

are analytic in the stars S_a, S_b respectively then the composite function

$$h(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$$

is analytic in the composite star $S_a \times S_b$. If in polar coördinates (r, t), $0 \leqq t < 2\pi$, the vertices of S_a and S_b are given by $r = A(t)$ and $r = B(t)$ then the vertices $r = C(t)$ of the composite star are defined by

$$C(t) = \inf_\tau A(\tau)B(t - \tau).$$ [1]

The transformation $z = e^{-s}$, $s = \sigma + it$ transforms a star into a domain of the form

$$\varphi(t) < \sigma < \infty \qquad -\infty < t < \infty.$$

For arbitrary $\varphi(t)$ the latter domain will be denoted by $D(\varphi)$. The composite of domains $D(\varphi)$, $D(\psi)$ is a domain $D(\chi)$, and

$$\chi(t) = \sup_\tau \{\varphi(\tau) + \psi(t - \tau)\}. \tag{1}$$

For Dirichlet series in general it is no longer true that if the functions

$$F(s) = \sum a_n e^{-\lambda_n s}, \qquad G(s) = \sum b_n e^{-\lambda_n s} \qquad (\lambda_n \geqq 0)$$

are analytic in $D(\varphi)$ and $D(\psi)$ then the function

$$H(s) = \sum a_n b_n e^{-\lambda_n s} \tag{2}$$

exists and is analytic in $D(\chi)$. The problem was investigated by Mandelbrojt[2] and he has established comprehensive results.

In the present note we will give another type of generalization of Hadamard's theorem. In some respects it will be less comprehensive than that of Mandelbrojt, however it will have the advantage of closely resembling the original

[1] This is a well known theorem of Hadamard. For a proof see e.g. S. Bochner and W. T. Martin, *Singularities of composite functions in several variables*, Annals of Math. 38 (1937), pp. 293–302, Annex.

[2] See Vladimir Bernstein, *Series de Dirichlet*, 1933, Chapitre VIII.

theorem on power series and referring to almost periodic functions[3] in general rather than to functions whose Dirichlet exponents λ_n form a monotone sequence.

In what follows every function of the real variable $t(\tau$, etc.) will be assumed, or constructed to be almost periodic (in the original definition of H. Bohr). For any $\varphi(t)$ we will say that the function $F(s)$ belongs to class C_φ if it is analytic and bounded in $D(\varphi)$ and almost periodic in some half-plane $D(\alpha)$, α = constant. With these definitions the following theorem holds.

THEOREM. *If $F(s)$ belongs to C_φ and $G(s)$ belongs to C_ψ then $H(s)$ exists and belongs to $C_{\chi+\delta}$ for any positive constant δ.*

The proof will consist of several steps.

1. The existence of $H(s)$ is quite easy to demonstrate. It can be constructed as a line "integral" by either expression

$$M_\tau\{F(\delta + i\tau)G(s - \delta - i\tau)\} \qquad (\delta > \sup \varphi(\tau))$$

$$M_\tau\{F(s - \delta - i\tau)G(\delta + i\tau)\} \qquad (\delta > \sup \psi(t))$$

the symbol $M_\tau h(\tau)$ denoting as usual the mean value of the given almost periodic function $h(\tau)$. The almost periodic nature, and the form (2) of the Dirichlet series of the function $H(s)$ can be easily derived from either expression if we approximate either $F(s)$ or $G(s)$ by an exponential polynomial.

2. We will use the symbol $D(\varphi; \psi)$ to denote the domain

$$\varphi(t) < \sigma < \psi(t), \qquad -\infty < t < \infty;$$

thus the previously defined symbol $D(\varphi)$ takes the place of $D(\varphi; \infty)$. Our first assertion is that if $F(s)$ is analytic and bounded in $D(\varphi)$ then for any constant $\rho > 0$ and any constant $\alpha > \rho + \sup \varphi(t)$ the function $F(s)$ is continuous uniformly for $s \subset D(\varphi + \rho; \alpha)$ and for $| F(s) | \leqq 1$.

If the assertion were false there would exist a sequence $\{F_n(s)\}$ and two sequences of points $\{a_n\}$ and $\{b_n\}$ in $D(\varphi + \rho; \alpha)$ such that $a_n - b_n$ tends to 0 but $F_n(a_n) - F_n(b_n)$ does not tend to 0. Putting $a_n = p_n + iq_n$ we may assume (replacing the sequence of indices $\{n\}$ by a subsequence) that the sequence of functions $\{\varphi(t + q_n)\}$ converges uniformly to a limit function $\varphi^*(t)$ and we may assume that for all n and t,

$$| \varphi(t + q_n) - \varphi^*(t) | < \rho/3.$$

Thus the functions

$$P_n(s) = F_n(s + iq_n)$$

would be analytic and uniformly bounded in $D(\varphi^* + \rho/3)$, but for a sequence of points $a_n' = p_n$, $\varphi^* + 2\rho/3 \leqq p_n \leqq \alpha$, and another sequence of points b_n' the difference $a_n' - b_n'$ would tend to 0 but not the difference $P_n(a_n') - P_n(b_n')$. This however is a contradiction to the fact that a family of functions which are

[3] For almost periodic functions of real and complex variables see A. S. Besicovitch, *Almost periodic functions*, 1932,

uniformly bounded in a domain are uniformly continuous in every compact subset of the domain.

3. Furthermore, corresponding to any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \varphi, \rho, \alpha)$ such that the inequality

$$| F(\alpha + it) | \leqq \delta$$

implies the inequality

$$| F(s) | \leqq \epsilon \qquad \text{for } s \subset D(\varphi + \rho; \alpha).$$

Otherwise there would exist a sequence $\{F_n(s)\}$ and a sequence of points $\{a_n\}$ in $D(\varphi + \rho; \alpha)$ such that

$$\sup_t | F_n(\alpha + it) | \to 0,$$

but $F_n(a_n)$ does not tend to 0. Introducing all symbols as in 2, the sequence $P_n(s)$ would be uniformly bounded in the domain $D(\varphi^* + \rho/3)$, convergent to 0 on the line $\sigma = \alpha$, but not uniformly convergent to 0 on a compact set of the latter domain, and this is a contradiction to Vitali's theorem.

4. If $F(s)$ belongs to C_φ, if ρ is a positive constant, and if $\lambda(t) \geqq \varphi(t) + \rho$ then the function

$$f_\lambda(t) = F(\lambda(t) + it)$$

is almost periodic. If $\{\lambda(t)\}$ is a set of functions belonging to a normal (that is compact) family[4] then the corresponding family $\{f_\lambda(t)\}$ is again normal. We have to show that each sequence $\{\tau_n\}$ contains a subsequence $\{t_n\}$ such that the sequence of functions $f_\lambda(t + t_n)$ is convergent, uniformly in all t and all λ. We first choose $\{t_n\}$ in such a way that

$$| \varphi(t + t_n) - \varphi^*(t) | < \rho/3$$

holds. Then the functions $F_n(s) = F(s + it_n)$ are analytic and uniformly bounded in $D(\varphi^* + \rho/3)$. $F(s)$ being almost periodic for $\sigma = \alpha$ we may now require from t_n that the sequence $F_n(s)$ shall converge uniformly in t for $\sigma = \alpha$. Denoting the limit function by $F^*(s)$ and denoting $\lambda(t + t_n)$ by $\lambda_n(t)$ we have

$$\begin{aligned} f_\lambda(t + t_m) - f_\lambda(t + t_n) &= F_m(\lambda_m + it) - F_n(\lambda_n + it) \\ &= \{F_m(\lambda_m + it) - F^*(\lambda_m + it)\} + \{F^*(\lambda_m + it) - F^*(\lambda_n + it)\} \\ &\quad + \{F^*(\lambda_n + it) - F_n(\lambda_n + it)\}. \\ &= R_1 + R_2 + R_3 . \end{aligned}$$

Applying 3 to the functions $F^* - F_m$ and $F^* - F_n$ we see that R_1 and R_3 tend uniformly to 0 as $m, n \to \infty$. Similarly, applying 2 to F^* we see that R_2 tends uniformly to 0 if we impose the further requirement on $\{t_n\}$ that the sequence $\{\lambda(t + t_n)\}$ shall be convergent uniformly in all t and all λ. This completes the proof of our statement.

[4] S. Bochner, *Beitrage zur theorie der fastperiodischen Funktionen*, Math. Annalen, 96 (1926), 119–147, 5.

5. The symbols φ, ψ, χ being those introduced in the theorem, if

$$\varphi(t) + \rho < \varphi_0(t) < \varphi(t) + 2\rho, \tag{3}$$

if $\varphi_0(t)$ has an almost periodic derivative, and if S is a compact set in $D(\chi + 3\rho)$ then the family of functions

$$h_s(\tau) = F(\varphi_0(\tau) + i\tau)\cdot G(s - \varphi_0(\tau) - i\tau) \qquad s \subset S$$

is a normal class. In fact, writing $s = \sigma + it$ and $\lambda_s(\tau) = \sigma - \varphi_0(t - \tau)$ we have

$$G(s - \varphi_0(\tau) - i\tau) = G(\lambda_s(t - \tau) + i(t - \tau)).$$

However

$$\lambda_s(\tau) = \sigma - \varphi_0(t - \tau) \geqq \chi(t) + 3\rho - \varphi_0(t - \tau)$$

$$\geqq \varphi(t - \tau) + \psi(\tau) + 3\rho - \varphi_0(t - \tau) \geqq \psi(\tau) + \rho$$

and our assertion follows from 4.

In particular we conclude that the limit as $T \to \infty$ of the expression

$$H_T(s) = \frac{-i}{2T}\int_{-T}^{T} F(\varphi_0(\tau) + i\tau)\cdot G(s - \varphi_0(\tau) - i\tau)\cdot\left(\frac{d\varphi_0(\tau)}{d\tau} + i\right) d\tau$$

exists uniformly for s in S and thus defines a function which is analytic and obviously also bounded in $D(\chi + 3\rho)$.

Now the last integral is the curve integral

$$\int F(\zeta)G(s - \zeta)\, d\zeta$$

taken along the path $\zeta = \varphi_0(t) + it$ from $t = -T$ to $t = T$. By Cauchy's theorem it is, for $\sigma > \alpha$, the sum of the integral

$$\int_{-T}^{T} F(\alpha + i\tau)G(s - \alpha - i\tau)\, d\tau$$

and of the expression

$$\int_{\varphi_0(-T)}^{\alpha} F(x - iT)G(s - x + iT)\, dx - \int_{\varphi_0(T)}^{\alpha} F(x + iT)G(s - x - iT)\, dx.$$

Now, the last expression is bounded and therefore, for $\sigma > \alpha$,

$$\lim_{T\to\infty} H_T(s) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(\alpha + i\tau)G(s - \sigma - i\tau)\, d\tau$$

and the right side is our function $H(s)$ as we have shown in 1.

6. Thus our function $H(s)$ is analytic and bounded in $D(\chi + 3\rho)$ if there exists a function $\varphi_0(t)$ which has an almost periodic derivative and for which (3) holds. Now, given $\varphi(t)$ and $\rho > 0$ we can approximate the function $\varphi(t) + 3\rho/2$ by an exponential polynomial $\varphi_0(t)$ to within $\rho/2$, and this completes the proof of our theorem.

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ON METRIC ARCS OF VANISHING MENGER CURVATURE[1]

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Introduction

1. Let Γ be a metric space which is a simple arc, that is the topological image of a closed linear segment. Menger introduced the following purely metric definition of curvature ([6], pp. 480, 481).[2] Let q, r, s be any distinct points of Γ. In virtue of the triangle inequality, their mutual distances qr, rs, qs are equal to the sides of a certain euclidean triangle. Let $\rho(q, r, s) = 1/\kappa(q, r, s)$ $(0 < \rho \leqq \infty, 0 \leqq \kappa < \infty)$ denote the radius of the circle circumscribed to that triangle, hence $\kappa(q, r, s)$ is its curvature. In fact

$$\begin{aligned}(1)\qquad &\kappa(q, r, s)\\ &= \frac{1}{qr\cdot rs\cdot qs}\{(qr + rs + qs)(qr + rs - qs)(qr - rs + qs)(-qr + rs + qs)\}^{1/2}.\end{aligned}$$

The space Γ is said to have in its point p the curvature $\kappa(p)$, provided

$$\lim \kappa(q, r, s) = \kappa(p)$$

as the distinct points q, r, s converge independently and simultaneously to p (in the sense of the metric of Γ). If the arc Γ is an ordinary straight segment we have $\kappa(q, r, s) = 0$ identically. The converse is also true and easy to prove. Of course, $\kappa(q, r, s) \equiv 0$ implies $\kappa(p) = 0$ in all points of Γ. Menger has raised the question whether the local limiting condition $\kappa(p) = 0$, in all points of Γ, does insure that Γ is congruent to a segment. Menger's negative answer to this question is already justified in view of the following trivial example: Let Γ be an arc of a circle exceeding half the circumference and metrised with the shorter arc length. As any sufficiently small subarc is congruent to a segment we have $\kappa(p) \equiv 0$ without the arc as a whole being congruent to a segment.

Menger's investigation of additional conditions on Γ, which will insure that vanishing curvature should imply straightness of the arc, have led him to the following theorem ([6], pp. 488–492):

Theorem 1 (of Menger). *A simple metric arc Γ is congruent to a segment if and only if (α) Γ has vanishing curvature in all its points, (β) Γ enjoys the four-point property.*

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[1] Presented to the American Mathematical Society, December 27, 1939.

[2] Concerning the subject of curvature in metric spaces see Blumenthal's book [3], Chapter V. Numbers in brackets refer to the list of references at the end of this paper.

The additional four-point property (β) means that any four points of Γ are congruent with some four points of the euclidean space E_3.

In the proof of Theorem 1, Menger uses the following proposition ([6], p. 487) which will be referred to as the n-lattice theorem for metric arcs: *For any positive integer n, there exists an ordered subset* $p_0, p_1, \cdots, p_n$, *of* $n + 1$ *points of* Γ, *with* p_0, p_n *coinciding with the endpoints of the arc* Γ *and such that* $p_0p_1 = p_1p_2 = \cdots = p_{n-1}p_n$. A set of such points p_ν is called a n-lattice inscribed in Γ. As pointed out by F. Alt and G. Beer ([1]), Menger's proof of the n-lattice theorem is inadequate, and they supplied a proof for the existence of n-lattices, provided the arc Γ belongs to, or may be imbedded in a euclidean space. With Alt and Beer's proof of the n-lattice theorem for euclidean arcs, Menger's proof of Theorem 1 became effective, provided condition (β) be replaced by the stronger one that Γ is a euclidean arc.

2. The object of this paper is twofold. *Firstly,* we shall prove the n-lattice theorem for arcs Γ which are not necessarily metric, but only semimetric with a continuous metric (§I below).[3] Hence Menger's proof of Theorem 1 becomes now fully effective. *Secondly,* we propose to relax Menger's condition (β) on the arc Γ still further (§II below), as stated by the following theorem.

THEOREM 2. *A simple metric arc* Γ *is congruent to a segment if and only if* (α)Γ *has vanishing curvature in all its points,* (β') *Ptolemy's inequality holds throughout* Γ.

By *Ptolemy's inequality* we mean the relation

$$ab\cdot cd + ad\cdot bc \geqq ac\cdot bd. \tag{2}$$

Condition (β') of Theorem 2 requires that (2) should be verified by the mutual distances of any four points a, b, c, d of Γ. This clearly amounts to the more symmetrical statement that for any four points, the three products of crosswise distances

$$ab\cdot cd, \quad ac\cdot bd, \quad ad\cdot bc$$

be equal to the sides of a certain euclidean triangle.

Let us say that a space is *ptolemaic* if the inequality (2) holds throughout the space. It is known that the euclidean plane E_2 is ptolemaic. But also E_3 is ptolemaic and therefore also E_m ($m \geqq 1$) as well as Hilbert space.[4] The triangle inequality $ab + bc \geqq ac$, that is, the metricity of a space, neither implies nor

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[3] I am indebted to Professor L. M. Blumenthal for calling my attention to the problem of proving the n-lattice theorem for metric arcs and for a stimulating correspondence during the preparation of this paper.

[4] In R. A. Johnson's book [5], p. 64, is found an elegant proof (by inversion) of Ptolemy's inequality in the plane. In Johnson's notation we always have $A'C' \leqq A'B' + B'C'$, whence $AB\cdot CD + AD\cdot BC \geqq AC\cdot BD$. The proof remains unchanged if A, B, C, D are any distinct points in E_3; in this case we perform an inversion with respect to a *sphere* of center D.

is it implied by Ptolemy's inequality (2). On the other hand the four-point property clearly implies (2), since E_3 is ptolemaic. The converse, however, is not true. Thus the quadruple of points a, b, c, d with the distances

$$ab = ad = bc = bd = cd = 1, \qquad ac = 2,$$

is metric and ptolemaic, but not euclidean. It is thus seen that the requirement "metric and ptolemaic" is not as strong as the requirement of the four-point property.

3. In concluding this introduction we shall discuss further examples of non-curved and non-straight arcs which might contribute to clarify the contents of Theorems 1 and 2. An example due to Menger is the real interval $-1 \leqq x \leqq 1$ metrised as follows: $xy = |x - y|$ if $x \cdot y \geqq 0$, $xy = |x| + |y| - x^2y^2$ if $x \cdot y < 0$. In spite of its having everywhere zero curvature, no subarc $-\epsilon \leqq x \leqq \epsilon$, no matter how small, is metrically a segment ([6], pp. 483–485).

A wide class of arcs of this type is readily found as follows. Let the function $\phi(t)$, defined for $t \geqq 0$, be subject to the following conditions:

$$(3) \quad \begin{cases} \phi(t) > 0 \text{ if } t > 0; \phi(0) = 0; \phi(t) \text{ is continuous; if the positive numbers } a, b, \\ c \text{ are the three sides of a triangle also } \phi(a), \phi(b), \phi(c) \text{ are sides of a triangle.} \end{cases}$$

If M is a metric space of distance function pq, it will still be metric if this distance function pq be changed to $\phi(pq)$. The space M, with the new metric $\phi(pq)$, is called the *metric transform* of M by the transforming function $\phi(t)$ and denoted by $\phi(M)$. The conditions (3) are for example verified by any continuous $\phi(t)$ $(t \geqq 0)$ which is monotone increasing, concave downward (in the non-strict sense so as to include $\phi(t) \equiv t$) and vanishing at the origin ([2], p. 8, [8], p. 64).

In addition to (3) let $\phi(t)$ also satisfy the conditions

$$(4) \quad \phi'(0) = 1, \qquad \phi''(0) = 0, \qquad \phi'''(0) = 0,$$

which imply the existence of the derivatives $\phi'(t)$, $\phi''(t)$ in a certain neighborhood of the origin and their continuity at the origin. Now E_1 certainly has everywhere zero curvature. Consequently also the metric transform $\phi(E_1)$ will have zero curvature, the reason being that the new metric $\phi(|x - y|)$ so closely "osculates" the old metric $|x - y|$ by virtue of (4), that the value zero of the curvature is an invariant of our metric transformation.[5] Postponing a proof until §III, we state this result as

LEMMA 1. *If the function $\phi(t)$ $(t \geqq 0)$ satisfies the conditions* (3) *and* (4), *then the space $\phi(E_1)$, that is the real axis with the metric $\phi(|x - y|)$, has everywhere zero curvature.*

On the other hand our space $\phi(E_1)$ is congruent to a straight line if and only

[5] This is an extension of a remark by Wilson (loc. cit.) that if $\phi(t) = t + o(t)$, i.e., $\phi'(0) = 1$, then arc-length is left invariant by the metric transformation from M to $\phi(M)$.

if $\phi(t) \equiv t$. We have therefore obtained a large class of non-curved and non-straight arcs. Indeed: *Any* $\phi(t)$ *satisfying the conditions* (3) *and* (4) *without* $\phi(t)$ *being identically* $= t$, *gives rise to an arc* $\phi(E_1)$ *which has zero curvature* (Lemma 1) *without being congruent to a finite or infinite segment.* All these conditions are e.g. verified if $\phi(t)$ is monotone increasing, concave downward and satisfies (4), i.e. the graph of $u = \phi(t)$ having at least a third order contact at the origin with the graph of $u = t$, while $\phi(t) \not\equiv t$. A particular example is

$$\phi_0(t) = t/(1 + t^3)^{1/3}. \tag{5}$$

As there is no $\epsilon > 0$ such that $\phi_0(t) \equiv t$ if $0 \leqq t \leqq \epsilon$, we see that no subarc of $\phi_0(E_1)$ no matter how small, is congruent to a segment.

This discussion shows that the class of non-curved, non-straight metric arcs is very large. It also throws some light on the additional conditions (β), or (β'), which force non-curved arcs to be straight. Take e.g. the condition (β'), of Theorem 2, that the space $\phi(E_1)$ be ptolemaic. In terms of $\phi(t)$, this amounts to the identical inequality

$$\phi(x)\phi(| y - z |) + \phi(y)\phi(| x - z |) \geqq \phi(z)\phi(| x - y |), \qquad (x, y, z \geqq 0). \tag{6}$$

Now the class of functions $\phi(t)$ satisfying (3) *and* (6) is still rather large. For example, it certainly contains all functions $\phi(t)(\not\equiv 0)$ such that $\phi(E_2)$, the metric transform of E_2 by $\phi(t)$, be congruently imbeddable in Hilbert space.[6] However, if we now require *in addition* that $\phi'(0) = 1$, $\phi''(0) = 0$, $\phi'''(0) = 0$, then we necessarily have $\phi(t) \equiv t$ for $0 \leqq t < \infty$. Let us state this consequence of Theorem 2 as

COROLLARY 1. *A function* $\phi(t)$ $(t \geqq 0)$ *satisfying the conditions* (3), (6), *and* (4), *reduces identically to* $\phi(t) = t$.

For example, $\phi_0(t)$ of (5) above, does satisfy (3) and (4). As $\phi_0(t) \not\equiv t$ we conclude that it violates (6).

A similar analytical statement, this time fully equivalent to Theorem 2, may be obtained in terms of a *general* distance function $f(x, y) = xy$ used to metrise the real interval $0 \leqq x \leqq 1$. It contains analogous local and global conditions which are necessary and sufficient in order that $f(x, y)$ be an *additive* function of the interval (x, y).

I. The n-lattice theorem for semimetric arcs

4. Let S be a semimetric space, that is a set of points $p, p', \cdots$, with a non-negative symmetric distance function pp' such that $pp' = 0$ if and only if

[6] As shown by von Neumann and Schoenberg, [7], the space $\phi(E_2)$ is imbeddable in Hilbert space, if and only if $\phi^2(t) = \int_0^\infty [1 - J_0(tu)]u^{-2}\, d\gamma(u)$, where J_0 is the Bessel function, $\gamma(u)$ is non-decreasing such that $\int_1^\infty u^{-2}\, d\gamma(u)$ exists. Such a $\phi(t)$ (if $\not\equiv 0$) vanishes only at the origin, hence it clearly satisfies the conditions (3). Hilbert space being ptolemaic, $\phi(t)$ also satisfies the inequality (6).

$p = p'$. We shall furthermore assume that the metric pp' is continuous, that is, the limiting relations $pp_n \to 0$, $p'p'_n \to 0$ shall imply that $p_np'_n \to pp'$.

In this section we establish the following theorem.

THEOREM 3. *Let S be a semimetric space with a continuous metric pp'. Let the point p_t, $(0 \leqq t \leqq 1)$, describe in S a continuous image Γ of the real interval $I: 0 \leqq t \leqq 1$. This image may have any number of multiple points, but we require that it have distinct end points p_0, p_1, hence*

$$p_0p_1 > 0. \tag{7}$$

Given a positive integer n and arbitrary positive numbers $\gamma_0, \gamma_1, \cdots, \gamma_n$, we can pick along our arc Γ such points

$$p_0, p_{t_1}, p_{t_2}, \cdots, p_{t_n}, p_1, \qquad (0 < t_1 < t_2 < \cdots < t_n < 1),$$

that their successive distances be all different from zero and proportional to $\gamma_0, \gamma_1, \cdots, \gamma_n$, that is

$$p_0p_{t_1}/\gamma_0 = p_{t_1}p_{t_2}/\gamma_1 = \cdots = p_{t_n}p_1/\gamma_n. \tag{8}$$

In case all the γ_i's are equal, this statement implies the $(n + 1)$-lattice theorem.

For convenience we set

$$f(t, s) = p_tp_s, \qquad (0 \leqq t \leqq s \leqq 1).$$

The continuity of p_t and the continuity of the metric of S readily imply that our function $f(t, s)$ is a continuous function of the point (t, s) in its triangular domain of definition $0 \leqq t \leqq s \leqq 1$. Moreover we have

$$f(t, s) \geqq 0, \quad f(t, t) = 0, \quad f(0, 1) > 0, \tag{9}$$

in view of (7). Setting

$$x_0 = t_1, \quad x_1 = t_2 - t_1, \cdots, x_n = 1 - t_n,$$

we see that the requirement (8) of the theorem amounts to proving the existence of numbers $x_0, x_1, \cdots, x_n$, such that

$$x_0 > 0, \quad x_1 > 0, \cdots, x_n > 0, \quad x_0 + x_1 + \cdots + x_n = 1, \tag{10}$$

and

$$\begin{aligned} f(0, x_0)/\gamma_0 = f(x_0, x_0 + x_1)/\gamma_1 &= f(x_0 + x_1, x_0 + x_1 + x_2)/\gamma_2 \\ &= \cdots = f(x_0 + \cdots + x_{n-1}, 1)/\gamma_n. \end{aligned} \tag{11}$$

Consider now in the $(n + 1)$-dimensional euclidean space E_{n+1} referred to cartesian coördinates $x_0, x_1, \cdots, x_n$, the closed simplex $\bar{\sigma}_n$ defined by the relations

$$\bar{\sigma}_n : x_0 \geqq 0, \quad x_1 \geqq 0, \cdots, x_n \geqq 0, \quad x_0 + x_1 + \cdots + x_n = 1. \tag{12}$$

Denote by P a generic point of $\bar{\sigma}_n$ of coördinates $(x_0, x_1, \cdots, x_n)$. Let us now map the simplex $\bar{\sigma}_n$ in itself by means of the transformation $P' = \Phi P$ explicitly defined as follows:

$$(13) \qquad \begin{aligned} &P' = \Phi P\colon x_0' = f(0, x_0)/N, \\ &x_1' = f(x_0, x_0 + x_1)/N, \cdots, x_n' = f(x_0 + \cdots + x_{n-1}, 1)/N, \end{aligned}$$

with $N = f(0, x_0) + f(x_0, x_0 + x_1) + \cdots + f(x_0 + \cdots + x_{n-1}, 1)$. We want to show that *the mapping $P \to P' = \Phi P$ is a single-valued and continuous mapping of $\bar{\sigma}_n$ in itself, i.e. the function ΦP is defined and continuous throughout $\bar{\sigma}_n$ and $\Phi P \epsilon \bar{\sigma}_n$.* Of especial importance here is the fact that the common denominator N in (13) is a *positive* function of P throughout $\bar{\sigma}_n$. Indeed, it is clear that $N \geqq 0$. But $N = 0$ implies $f(0, x_0) = f(x_0, x_0 + x_1) = \cdots = f(x_0 + \cdots + x_{n-1}, 1) = 0$, hence $p_0p_{t_1} = p_{t_1}p_{t_2} = \cdots = p_{t_n}p_1 = 0$, whence, since S is semimetric, $p_0 = p_{t_1} = \cdots = p_1$. Thus $p_0 = p_1$, or $p_0p_1 = 0$ and this contradicts our assumption (7). Thus $P \epsilon \bar{\sigma}_n$ implies $N = N(P) > 0$. As $N(P)$ is a continuous function in $\bar{\sigma}_n$ and $\bar{\sigma}_n$ being compact and closed, we conclude that $N(P)$ is uniformly bounded away from zero throughout $\bar{\sigma}_n$. Now it is clear that $P' = \Phi P$ is well-defined and continuous in $\bar{\sigma}_n$.

In addition, our mapping $P \to P' = \Phi P$ has a further property which is to be discussed now. Let $A_0, A_1, \cdots, A_n$ be the vertices of σ_n, that is the points of coördinates $(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$.

If $\bar{\sigma}_k = [A_{i_0}, A_{i_1}, \cdots, A_{i_k}]$, $(0 \leqq i_0 < i_1 < \cdots < i_k \leqq n; 0 \leqq k \leqq n)$, represents the k-dimensional closed simplex of vertices $A_{i_0}, \cdots, A_{i_k}$, that is a closed k-face of $\bar{\sigma}_n$, then

$$(14) \qquad P \epsilon \bar{\sigma}_k \quad \textit{implies that} \quad P' = \Phi P \epsilon \bar{\sigma}_k.$$

In other words, our mapping $P \to P'$ maps every $\bar{\sigma}_k$ in itself, where $k = 0, 1, \cdots,$ or n.

Indeed true, for the requirement $P \epsilon \bar{\sigma}_k$ where $P = (x_0, \cdots, x_n)$, is equivalent (besides $\sum x = 1$) to the relations

$$x_\nu = 0 \quad \text{if} \quad \nu \neq i_0, i_1, \cdots, i_k, \qquad x_\nu \geqq 0 \quad \text{if} \quad \nu = i_0, i_1, \cdots, i_k.$$

But then (13) likewise implies that

$$x_\nu' = 0 \quad \text{if} \quad \nu \neq i_0, i_1, \cdots, i_k, \qquad x_\nu' \geqq 0 \quad \text{if} \quad \nu = i_0, i_1, \cdots, i_k,$$

since $x_\nu = 0$ implies that $x_\nu' = 0$. Hence $P' \epsilon \bar{\sigma}_k$.

We need now the following topological proposition which we state as

Lemma 2. *The property (14) of our mapping $P \to P'$ implies that every point R of $\bar{\sigma}_n$ is an image point of this mapping. In symbols this means:*

$$\Phi\bar{\sigma}_n \supset \bar{\sigma}_n, \quad \textit{hence} \quad \Phi\bar{\sigma}_n = \bar{\sigma}_n.$$

We shall return to this lemma in a moment. Granted its truth it is now clear how the proof of Theorem 3 is to be completed. Consider the point R

where the half-line OR of direction parameters $(\gamma_0, \cdots, \gamma_n)$ meets the hyperplane $x_0 + \cdots + x_n = 1$. R is a point of σ_n and its coördinates are

$$x_0' = \gamma_0 \Big/ \sum_0^n \gamma_\nu, \qquad x_1' = \gamma_1 \Big/ \sum_0^n \gamma_\nu, \cdots, x_n' = \gamma_n \Big/ \sum_0^n \gamma_\nu. \tag{15}$$

Hence, by Lemma 2, R is the image of some point $P = (x_0, \cdots, x_n)$ of $\bar{\sigma}_n$. But then, by equating the values of the x_ν''s given by (13) and (15), we obtain the desired relations (11), hence (8). Notice that the inequalities (10) also follow as a consequence.

5. In order to prove Lemma 2, let us consider our simplex $\bar{\sigma}_n = [A_0, \cdots, A_n]$ as imbedded in the space E_n defined by $x_0 + \cdots + x_n = 1$. We shall operate now in this E_n exclusively. Suppose that to every point P of $\bar{\sigma}_n$ there is attached a vector $\boldsymbol{PQ}$ ($P, Q \in E_n$) which varies continuously with P. If for every point P on the boundary of $\bar{\sigma}_n$ the corresponding vector $\boldsymbol{PQ}$ *points into the interior of* $\bar{\sigma}_n$, then it is known that this vector field has at least one singular point in σ_n, that is, a point where $\boldsymbol{PQ} = 0$ ($P = Q$). By "pointing into the interior" we mean that there is on $\boldsymbol{PQ}$ a point P_1, between P and Q, such that the open segment $\overline{PP_1}$ lies in σ_n.[7]

This classical result implies our Lemma 2 without difficulty. Indeed, let R_0 be a point of $\bar{\sigma}_n$ which is not an image point of the mapping $P \to P'$. Without restricting the generality we may assume that $R_0 \in \sigma_n$, for if $R_0 \in \bar{\sigma}_n - \sigma_n$, then $R_0 \in \sigma_k$, σ_k being some k-face of σ_n. The case $k = 0$ being obviously impossible, we may assume $1 \leqq k \leqq n - 1$. Since $\Phi\bar{\sigma}_k \subset \bar{\sigma}_k$ (property (14)) we may then repeat the present argument with $\bar{\sigma}_k$ instead of $\bar{\sigma}_n$. We now construct a vector field as follows. Attach to P the vector $\boldsymbol{PQ} = \boldsymbol{P'R_0}$. This vector $\boldsymbol{PQ}$ is a continuous vector function of P which never vanishes (since P' never $= R_0$).

It remains to show that along the boundary of $\bar{\sigma}_n$, $\boldsymbol{PQ}$ points into the interior of $\bar{\sigma}_n$. Let P be on the boundary of $\bar{\sigma}_n$ and $P \in \sigma_k$ $(0 \leqq k \leqq n - 1)$, hence also $P' \in \bar{\sigma}_k$. We may evidently choose in E_n an oblique system of axes such that the defining relation of our simplices $\bar{\sigma}_n$ and σ_k are

$$\bar{\sigma}_n : y_1 \geqq 0, \cdots, y_n \geqq 0, \quad y_1 + \cdots + y_n \leqq 1,$$
$$\sigma_k : y_1 > 0, \cdots, y_k > 0, \quad y_{k+1} = \cdots = y_n = 0, \quad y_1 + \cdots + y_n < 1. \tag{16}$$

[7] This well known result is most readily derived from Hadamard's note [4] as follows. Choose a cartesian coördinate system in E_n. Let the equations

$$\boldsymbol{PQ} = (f_1(P), f_2(P), \cdots, f_n(P)), \qquad \boldsymbol{PI} = (g_1(P), g_2(P), \cdots, g_n(P)),$$

indicate the components of the vector functions $\boldsymbol{PQ}$, $\boldsymbol{PI}$, where I is a certain fixed point of σ_n. Let us assume that $\boldsymbol{PQ}$ never vanishes, hence the $f_\nu(P)$ never vanish simultaneously in $\bar{\sigma}_n$. This implies that the system $\{f_\nu(P)\}$ has zero index on the boundary of $\bar{\sigma}_n$ (loc. cit. p. 466). On the other hand, the index of the system $\{g_\nu(P)\}$ is clearly ± 1. Hence, by the Poincaré-Bohl theorem (p. 467), we conclude to the existence of a point P on the boundary of $\bar{\sigma}_n$, such that $\boldsymbol{PQ}$ and $\boldsymbol{PI}$ lie on the same line and point in opposite directions. As $\boldsymbol{PI}$ points into the interior, $\boldsymbol{PQ}$ would point into the exterior, against our assumption.

Let $P = (y_1, \cdots, y_n)$, $R_0 = (\eta_1, \cdots, \eta_n)$, hence all η_ν's are positive and the y_ν's satisfy the relations (16). Let $P' = (y_1', \cdots, y_n')$. Since $P' \in \bar{\sigma}_k$ we have $y_{k+1}' = \cdots = y_n' = 0$. Now $\boldsymbol{PQ} = \boldsymbol{P'R}_0 = (\eta_1 - y_1', \cdots, \eta_n - y_n')$. Let T be a point on $\boldsymbol{PQ}$ of coördinates

$$t_\nu = y_\nu + \rho(\eta_\nu - y_\nu'), \qquad (0 < \rho < 1).$$

By (16) we have $t_1 > 0, \cdots, t_k > 0$, for sufficiently small values of ρ. However, if $\nu > k$ we have $t_\nu = y_\nu + \rho(\eta_\nu - y_\nu') = \rho\eta_\nu > 0$, in any case. Finally $t_1 + \cdots + t_n = \sum y_\nu + \rho \sum \eta_\nu - \rho \sum y_\nu'$ is <1, for sufficiently small ρ, in view of the last inequality (16). Thus $T \in \sigma_n$, provided ρ is positive and sufficiently small. This concludes the proof of Lemma 2 and Theorem 3 is established.

6. We conclude the discussion of the n-lattice theorem with two remarks concerning some of the assumptions occurring in Theorem 3.

i. Suppose that we drop the assumption (7), everything else unchanged, and assume our curve Γ to be closed ($p_0 = p_1$, $p_t \neq p_0$). Ruling out the now always present trivial solution of a lattice with all sides equal to zero, the problem would be to show that there exists a subdivision $0 < t_1 < \cdots < t_n < 1$, such that $p_0p_{t_1} = p_{t_1}p_{t_2} = \cdots = p_{t_n}p_1 > 0$. However, such a solution need not always exist. Indeed, consider in E_1 the curve defined as follows: $x = t$ if $0 \leqq t \leqq \frac{1}{2}$, $x = 1 - t$ if $\frac{1}{2} \leqq t \leqq 1$, i.e. a closed polygon of two sides. Here there is obviously no non-trivial lattice of three sides ($n = 2$), nor of any odd number of sides. Such non-trivial lattices presumably always exist, provided our closed curve Γ is *simple*, i.e. the topological image in S of a circle. Our proof, however, does not cover this case, being essentially based on the assumption (7). Thus our method establishes the existence of n-lattices inscribed in open space-filling curves, but does not prove the existence of an equilateral triangle inscribed in a given ellipse with one vertex in a given point of the curve.

ii. The assumption of the continuity of the metric of S is not only essential for our proof, but without it Theorem 3 breaks down. To show this, let us consider the arc Γ: $p_t = t$ $(0 \leqq t \leqq 1)$ with the following metric: let $p_tp_s = |t - s|$, with the sole exception that if $\frac{1}{2} \leqq t \leqq 1$, we set $p_tp_1 = p_1p_t = (1 - t)/2$ (instead of $1 - t$). Γ *is a semimetric space,* since $p_tp_s = 0$ implies $t = s$. Moreover, Γ *is an open simple arc,* since $p_0p_1 = 1$, while $t_n \to t$ implies $p_{t_n}p_t \to 0$. *There is no 2-lattice inscribed in* Γ, *that is there is no value of t such that* $p_0p_t = p_tp_1$. For if $0 \leqq t < \frac{1}{2}$ we have $p_0p_t = t < 1 - t = p_tp_1$, while if $\frac{1}{2} \leqq t \leqq 1$ we have $p_0p_t = t \geqq \frac{1}{2} > (1 - t)/2 = p_tp_1$. The reason for this breakdown must naturally lie in the *discontinuity of the metric,* in view of Theorem 3. This discontinuity is apparent, for if $t \to \frac{1}{2}$ $(t < \frac{1}{2})$ we have $p_tp_1 \to \frac{1}{2}$, while $p_{1/2}p_1 = \frac{1}{4}$.

II. On arcs of vanishing curvature in ptolemaic spaces

7. Menger's proof of Theorem 1 is based on two results: One is the n-lattice theorem which does hold, as we have just seen, in very general spaces. The

other result is a comparison theorem concerning finite sets of points in a space with the four-point property. We state it as follows in the special form actually used by Menger in his proof of Theorem 1.

LEMMA 3. *Let $p_0, p_1, \cdots, p_n$ be a set of $n+1$ points enjoying the four-point property and let $p_0', p_1', \cdots, p_n'$ be $n+1$ points of a plane E_2 such that*

$$(17)\qquad p_0p_1 = p_1p_2 = \cdots = p_{n-1}p_n = p_0'p_1' = p_1'p_2' = \cdots = p_{n-1}'p_n' > 0,$$

$$(18)\qquad p_0'p_2' \leqq p_0p_2, \qquad p_1'p_3' \leqq p_1p_3, \cdots, p_{n-2}'p_n' \leqq p_{n-2}p_n,$$

(19) *The points p_ν' lie on a circle K, of E_2, of finite or infinite radius, such that the polygon $p_0'p_1' \cdots p_n'p_0'$ is convex.*

These assumptions imply that

$$(20)\qquad p_0'p_n' \leqq p_0p_n.$$

This lemma is the only part of Menger's proof of Theorem 1 in which the four-point property intervenes. Theorem 2 will be therefore proved, provided we can justify the conclusion $p_0'p_n' \leqq p_0p_n$ of Lemma 3 under the weaker assumption that the set $p_0, p_1, \cdots, p_n$ is ptolemaic. This we state as

LEMMA 4. *The conclusion (20) of Lemma 3 is still valid if, everything else unchanged, we replace the assumption of the four-point property by the assumption that the set $p_0, p_1, \cdots, p_n$ is ptolemaic.*

PROOF. Let $p_0'p_1' = s_1$, $p_0'p_2' = s_2$, $\cdots$, $p_0'p_n' = s_n$. Since, in view of (17) and (19), $p_0'p_1' \cdots p_n'$ is a regular polygonal line, we now have

$$(21)\qquad p_i'p_{i+k}' = s_k, \qquad (0 \leqq i < i + k \leqq n).$$

By applying Ptolemy's relation to the convex quadruples $p_0', p_{\nu-2}', p_{\nu-1}', p_\nu'$ $(\nu = 3, 4, \cdots, n)$ inscribed in the circle K, we get among the s_k's the relations $s_1s_{\nu-2} + s_1s_\nu = s_2s_{\nu-1}$. Setting $a = s_2/s_1$, this gives the following recurrence relation

$$(22)\qquad s_{\nu-2} - as_{\nu-1} + s_\nu = 0, \qquad (\nu = 3, 4, \cdots, n).$$

Applying now Ptolemy's inequality (1) to the quadruple $p_0, p_{\nu-2}, p_{\nu-1}, p_\nu$, $(3 \leqq \nu \leqq n)$, we get

$$p_0p_{\nu-2}\cdot p_{\nu-1}p_\nu + p_0p_\nu\cdot p_{\nu-2}p_{\nu-1} \geqq p_0p_{\nu-1}\cdot p_{\nu-2}p_\nu.$$

Setting

$$(23)\qquad x_k = p_0p_k, \qquad (k = 0, 1, \cdots, n),$$

this gives $x_{\nu-2}s_1 + x_\nu s_1 \geqq x_{\nu-1}\cdot p_{\nu-2}\,p_\nu \geqq x_{\nu-1}s_2 = x_{\nu-1}as_1$. We thus have

$$(24)\qquad D^2x_{\nu-2} \equiv x_{\nu-2} - ax_{\nu-1} + x_\nu \geqq 0, \qquad (\nu = 3, 4, \cdots, n).$$

In view of (22) we get now the following identical relation among the x_ν's

$$s_{n-1}D^2x_0 + s_{n-2}D^2x_1 + \cdots + s_1D^2x_{n-2}$$
$$= s_1x_n + (s_{n-2} - as_{n-1})s_1 = s_1x_n + (-s_n)s_1,$$

whence

$$s_1(x_n - s_n) = s_{n-1}(x_2 - ax_1 + x_0) + s_{n-2}D^2x_1 + s_{n-3}D^2x_2 + \cdots + s_1D^2x_{n-2}.$$

But $ax_1 = (s_2/s_1)s_1 = s_2$, $x_0 = 0$. Hence

$$(25) \quad s_1(x_n - s_n) = s_{n-1}(x_2 - s_2) + s_{n-2}D^2x_1 + s_{n-3}D^2x_2 + \cdots + s_1D^2x_{n-2}.$$

This relation completes the proof that $p_0p_n - p_0'p_n' = x_n - s_n \geqq 0$, in view of the inequalities (24) and our assumption (18) which implies that $x_2 - s_2 \geqq 0$.[8]

We want to mention that (25) also implies the following: If $p_0p_n = p_0'p_n'$, hence $x_n - s_n = 0$, then $x_2 - s_2 = 0$ and all "second differences" D^2x_ν ($\nu = 1, \cdots, n-2$) must vanish, since all terms of the right-hand side of (25) are non-negative. We then have the equality sign in all relations (24). Now (22) and (24) show that $x_\nu = s_\nu$, or $p_0p_\nu = p_0'p_\nu'$ ($\nu = 1, \cdots, n$). Applying this argument repeatedly we conclude that our ptolemaic set $p_0, p_1, \cdots, p_n$ is congruent with the euclidean set $p_0', p_1', \cdots, p_n'$.

In conclusion it should be explicitly stated that only the special form Lemma 3 of Menger's *Lemma* ([6], p. 486 and pp. 491–492) allows the above replacement of the four-point property by the weaker assumption of Ptolemy's inequality. Menger's general Lemma allows no such weakening. This is readily shown by obvious examples.

III. A class of arcs of vanishing curvature

8. In this last section we prove the Lemma 1 of the Introduction to the effect that if the function $\phi(t)$ satisfies the conditions (3) and (4), then the space $\phi(E_1)$ has everywhere zero curvature.

Let x, y, z be three distinct points of $\phi(E_1)$ and let $x < y < z$. Let $y - x = a$, $z - y = b$, hence $z - x = a + b$. The sides of the triangle formed by the triple (x, y, z) are therefore $\phi(a), \phi(b), \phi(a+b)$. By (1) we have

$$(\kappa(x, y, z))^2 = \frac{\begin{array}{r}[\phi(a) + \phi(b) + \phi(a+b)][\phi(a) + \phi(a+b) - \phi(b)] \\ \cdot[\phi(b) + \phi(a+b) - \phi(a)][\phi(a) + \phi(b) - \phi(a+b)]\end{array}}{[\phi(a)\phi(b)\phi(a+b)]^2}$$

and we must show that this expression tends to zero as the positive numbers a, b tend to zero independently.

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[8] It should be noticed that the recurrence relation (22), if applied beyond $\nu = n$, gives rise to an infinite sequence of numbers $s_1, s_2, \cdots$, which are by no means all positive, if $a < 2$. Only $s_1, s_2, \cdots, s_{n-1}$, are certainly positive and $s_n \geqq 0$, on account of the convexity of the polygon $p_0'p_1' \cdots p_n'p_0'$. In case $a = 2$ (the circle K is a straight line) our discussion reduces to the following geometrically obvious situation: Consider in a plane coördinate system the polygonal line joining successively the points $(0, 0)$, $(1, x_1)$, $(2, x_2)$, $\cdots$, (n, x_n). If this polygonal line is convex downward, then $x_n \geqq nx_1$. Moreover, $x_n = nx_1$ implies that $x_\nu = \nu x_1$ ($\nu = 1, 2, \cdots, n$).

Since $\phi'(0) = 1$ we conclude that $\phi(a)/a \to 1, \phi(b)/b \to 1, \phi(a+b)/(a+b) \to 1$. It therefore suffices to prove that

$$\left[\frac{\phi(a)\phi(b)\phi(a+b)}{ab(a+b)}\right]^2 (\kappa(x, y, z))^2 = \frac{\phi(a) + \phi(b) + \phi(a+b)}{a+b}$$

$$\cdot \frac{\phi(a) + \phi(a+b) - \phi(b)}{a} \cdot \frac{\phi(b) + \phi(a+b) - \phi(a)}{b} \cdot \frac{\phi(a) + \phi(b) - \phi(a+b)}{ab(a+b)}$$

tends to zero with a and b. We now determine the limits of the four factors on the right-hand side.

In the first place we have

$$\frac{\phi(a) + \phi(b) + \phi(a+b)}{a+b} \to 2.$$

Indeed, by the mean-value theorem we get

$$\left|\frac{\phi(a) + \phi(b) + \phi(a+b)}{a+b} - 2\right|$$

$$= \left|\frac{a[\phi'(a\theta_1) - 1] + b[\phi'(b\theta_2) - 1] + (a+b)[\phi'(a\theta_3 + b\theta_3) - 1]}{a+b}\right|$$

$$\leqq |\phi'(a\theta_1) - 1| + |\phi'(b\theta_2) - 1| + |\phi'(a\theta_3 + b\theta_3) - 1|$$

and this last expression tends to zero with a and b since $\phi'(t) \to 1$ as $t \to 0$.

The next two factors converge likewise to the limit two:

$$\frac{\phi(a) + \phi(a+b) - \phi(b)}{a} \to 2, \qquad \frac{\phi(b) + \phi(a+b) - \phi(a)}{b} \to 2.$$

Indeed, for reasons of symmetry it suffices to prove the first relation. Again the mean-value theorem gives

$$\frac{\phi(a) + \phi(b+a) - \phi(b)}{a} = \frac{\phi(a) + a\phi'(b+\theta a)}{a} = \frac{\phi(a)}{a} + \phi'(b+\theta a) \to 1 + 1 = 2.$$

It therefore remains to prove that

$$\lim_{\substack{a\to 0\\ b\to 0}} \frac{\phi(a) + \phi(b) - \phi(a+b)}{ab(a+b)} = 0.$$

Indeed, by the symmetry in a and b we may assume $0 < a \leqq b$. Applying repeatedly the mean-value theorem we have

$$\frac{\phi(a) + \phi(b) - \phi(a+b)}{ab(a+b)} = \frac{\phi(a) - \phi(0) - [\phi(b+a) - \phi(b)]}{ab(a+b)}$$

$$= \frac{a\phi'(\xi) - a\phi'(\eta)}{ab(a+b)} = \frac{\phi'(\xi) - \phi'(\eta)}{b(a+b)} = \frac{|\xi - \eta| \cdot |\phi''(\zeta)|}{b(a+b)}$$

$$\leqq \frac{|\phi''(\zeta)|}{b} = \left|\frac{\phi''(\zeta) - \phi''(0)}{\zeta}\right| \cdot \frac{\zeta}{b}$$

and this last expression tends to zero with b, since

$$[\phi''(\zeta) - \phi''(0)]/\zeta \to \phi'''(0) = 0, \qquad \zeta/b < \eta/b < (a + b)/b \leqq 2b/b = 2.$$

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ON CONTINUITY AND PERIODICITY OF MEASURABLE FUNCTIONS[1]

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1. Introduction. Let $x(t)$ denote a complex-valued function defined for almost all t and measurable in the sense of Lebesgue over $-\infty < t < \infty$. In §2 we show (Theorem 1) existence of a decreasing sequence $H_1 > H_2 > \cdots$ converging to 0 with the following property: If $h_1, h_2, \cdots$ is a real sequence with $|h_n| < H_n$ for each $n = 1, 2, \cdots$ then for each t_0 with the exception of a null set (which may depend upon the particular sequence h_n) $x(t)$ is continuous at t_0 over the set $t_0, t_0 + h_1, t_0 + h_2, \cdots$. In §3 we give theorems to show that lim $x(t + h_n) = x(t)$ may fail for each t in a set of positive measure or for almost all t even when $x(t)$ belongs to a restricted class of functions and the sequence $h_1, h_2, \cdots$ is a decreasing sequence which converges to 0. In §4 we apply Theorem 1 to show that if $x(t)$ and $y(t)$ are measurable over $-\infty < t < \infty$, then

$$G(\lambda) = \underset{-\infty<t<\infty}{\text{m.u.b.}} |x(t - \lambda) - y(t)| \tag{1}$$

is a lower semi continuous function of λ, i.e. that

$$G(\lambda_0) \leqq \liminf_{\lambda\to\lambda_0} G(\lambda) \qquad -\infty<\lambda_0<\infty. \tag{2}$$

The measurable upper bound (m.u.b.) over $-\infty < t < \infty$ of a real measurable function $w(t)$ is the least number β, which may be $+\infty$, such that $w(t) \leqq \beta$ for almost all t. In §5 we give some results involving periodicity of measurable functions suggested by §4. While Theorem 4 of §4 is used at one point in the proof of Theorem 5, an alternative method of treatment is given which makes §5 completely independent of earlier sections.

2. Existence of the sequence H_n. We now prove the following theorem involving a function $x(t)$ which may be unbounded and non-integrable over each finite interval.

Theorem 1. *If $x(t)$ is measurable over $-\infty < t < \infty$, then there is a decreasing sequence $H_1, H_2, \cdots$ of positive numbers converging to 0 such that if $-H_n < h_n < H_n$ for each $n = 1, 2, \cdots$ then*

$$\lim_{n\to\infty} x(t + h_n) = x(t) \tag{3}$$

for almost all t.

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If $x(t)$ is measurable over $-\infty < t < \infty$, then the function $y(t)$ defined by

$$y(t) = \frac{1}{1+t^2} \frac{x(t)}{1+|x(t)|} \tag{4}$$

is measurable and $|y(t)| < 1/(1+t^2)$ so that $y(t)$ is integrable over $-\infty < t < \infty$. Hence

$$\lim_{h\to 0} \int_{-\infty}^{\infty} |y(t+h) - y(t)|\, dt = 0. \tag{5}$$

This implies existence of positive numbers $H_1 > H_2 > \cdots$ such that $H_n \to 0$ and

$$\int_{-\infty}^{\infty} |y(t+h_n) - y(t)|\, dt < 2^{-n} \qquad |h_n| < H_n. \tag{6}$$

Let a sequence $h_1, h_2, \cdots$ for which $|h_n| < H_n$ be fixed. The inequalities (6) and a fundamental theorem on termwise integration of series of non-negative terms imply that

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |y(t+h_n) - y(t)|\, dt = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |y(t+h_n) - y(t)|\, dt < 1, \tag{7}$$

the series

$$\sum_{n=1}^{\infty} |y(t+h_n) - y(t)| \tag{8}$$

being convergent for almost all t. For each t for which the series (8) converges, we have

$$\lim_{n\to\infty} y(t+h_n) = y(t); \tag{9}$$

hence (9) holds for almost all t. Since $h_n \to 0$, (9) and (4) imply (3). Therefore (3) holds for almost all t and Theorem 1 is proved.

The null set of t for which (3) fails to hold may depend upon the sequence h_n. This is true for the function $x_1(t)$ which is 1 when t is rational and 0 when t is irrational. In fact if $H_1, H_2, \cdots$ is any sequence of positive numbers and t' is fixed, then it is easy to show existence of a sequence $h_1', h_2', \cdots$ such that $0 < h_n' < H_n$ and

$$\limsup_{n\to\infty} |x_1(t'+h_n') - x_1(t')| = 1. \tag{10}$$

Thus when $x = x_1$, the set of t for which (3) holds for each sequence h_n with $0 < h_n < H_n$ is empty.

3. Sequence h_n not satisfying the condition $|h_n| < H_n$. The gist of Theorem 1 is that if $x(t)$ is measurable and h_n converges to 0 sufficiently rapidly, then $\lim x(t+h_n) = x(t)$ for almost all t. The hypothesis that $x(t)$ is measurable and h_n is a decreasing sequence of positive elements converging to 0 does not imply

that $\lim x(t + h_n) = x(t)$ for almost all t. This is a consequence of Theorem 2 and the fact that there exist non-dense sets of positive measure.

THEOREM 2. *If $\varphi(t)$ is the characteristic function of a non-dense set W, then there is a decreasing sequence $h_1, h_2, \cdots$ of positive numbers converging to* 0 *such that*

$$\limsup_{n\to\infty} | \varphi(t + h_n) - \varphi(t) | = 1 \qquad t \in W. \tag{11}$$

A set W is called non-dense when its closure $\overline{W}$ contains no intervals; and the characteristic function of W is the function $\varphi(t)$ for which $\varphi(t) = 1$ when $t \in W$ and $\varphi(t) = 0$ otherwise. The complement B of $\overline{W}$ is open and hence is the union of a countable set of non-overlapping open "black" intervals $B_1, B_2, \cdots$. For each $p = 1, 2, \cdots$ let I_p denote the interval $-p < t < p$ and let $C(p) = B_1 + B_2 + \cdots + B_p$. The fact that $\overline{W}$ is non-dense implies that for each $p = 1, 2, \cdots$ there is an index α_p such that each subinterval of I_{p+1} of length 2^{-p} contains at least one point of $C(\alpha_p)$. (It suffices to choose α_p so great that $C(\alpha_p)$ contains at least one point in each of the fixed set of subintervals of I_{p+1} of length 2^{-p-1} obtained by bisection of unit intervals and resulting subintervals.) Hence for each $p = 1, 2, \cdots$ there is a finite set $h_1^{(p)}, h_2^{(p)}, \cdots, h^{(p)}(\beta_p)$ of positive numbers, each less than 2^{-p}, such that if $t \in I_p - C(\alpha_p)$ then $x + h_n^{(p)} \in C(\alpha_p)$ for at least one n. (If r is positive and less than the length of the shortest of the α_p intervals in $C(\alpha_p)$, then a finite set of the numbers $r, 2r, 3r, \cdots$ will serve.) If $A > 0$, the set of all numbers $h_n^{(p)}$ greater than A is a finite set. Hence the set of all numbers $h_n^{(p)}$ can be arranged in a decreasing sequence converging to 0; let this sequence be denoted by $h_1, h_2, \cdots$.

Let $t \in W$ be fixed. Choose P such that $t \in I_p$ when $p > P$. Then $t \in I_p - C(\alpha_p)$ when $p > P$. Hence if $p > P$, then $t + h_n^{(p)} \in C(\alpha)$ for some n. But each number $h_n^{(p)}$ is an element of the sequence $h_1, h_2, \cdots$ between 0 and 2^{-p}; and $C(\alpha) \subset B$. This implies that if $p > P$, then $t + h_n \in B$ for some n such that $0 < h_n < 2^{-p}$. Therefore $t + h_n \in B$ for an infinite set of n. For each such n, we have

$$| \varphi(t + h_n) - \varphi(t) | = | 0 - 1 | = 1. \tag{12}$$

But $| \varphi(t + h_n) - \varphi(t) | \leqq 1$ for each $n = 1, 2, \cdots$. Therefore (11) holds and Theorem 2 is proved.

THEOREM 3. *If D is a set of the first category, then there exist a real bounded measurable function $\psi(t)$ and a decreasing sequence $h_1, h_2, \cdots$ of positive numbers converging to* 0 *such that*

$$\limsup_{n\to\infty} | \psi(t + h_n) - \psi(t) | > 0 \qquad t \in D. \tag{13}$$

The set D, being of the first category, is the union of a countable set of non-dense sets, say $W_1, W_2, W_3, \cdots$. For each $p = 1, 2, \cdots$ let $\varphi_p(t)$ be the characteristic function of $\overline{W}_p$. Let

$$\psi(t) = \sum_{p=1}^{\infty} 3^{-p} \varphi_p(t) \qquad -\infty < t < \infty. \tag{14}$$

Then the functions $\varphi_p(t)$ and $\psi(t)$ are measurable, non-negative, and $\leqq 1$. By Theorem 2 there is for each index $p = 1, 2, \cdots$ a decreasing sequence $h_1^{(p)}, h_2^{(p)}, \cdots$ converging to 0 such that

$$\limsup_{n\to\infty} | \varphi_p(t + h_n^{(p)}) - \varphi_p(t) | = 1 \qquad t \in \overline{W}_p ; \tag{15}$$

and it is obviously possible (simply by removing a finite number of the elements of the sequence $h_n^{(p)}$ and using the remaining sequence) to obtain $h_n^{(p)} < 2^{-p}$ for each $n = 1, 2, \cdots$. Let the set of all numbers $h_n^{(p)}, n, p = 1, 2, \cdots$, be arranged in a decreasing sequence which we denote by $h_1, h_2, \cdots$. Then for each $p = 1, 2, \cdots$

$$\limsup_{n\to\infty} | \varphi_p(t + h_n) - \varphi_p(t) | = 1 \qquad t \in \overline{W}_p . \tag{16}$$

But the definition (14) of $\psi(t)$ and the fact that $| \varphi_p(t + h_n) - \varphi_p(t) |$ is always 0 or 1 implies that

$$\lim_{n\to\infty} \psi(t + h_n) = \psi(t) \tag{17}$$

if and only if

$$\lim_{n\to\infty} \varphi_p(t + h_n) = \varphi_p(t) \qquad p = 1, 2, \cdots . \tag{18}$$

Since (16) implies that (18) fails for each $t \in \Sigma\overline{W}_p$, (17) fails and (13) holds for each $t \in \Sigma\overline{W}_p$ and *a fortiori* for each $t \in D = \Sigma W_p$. This completes the proof of Theorem 3.

Since the complement of a set D of the first category may be a null set (this will be the case if $D = W_1 + W_2 + \cdots$ where each non-dense set W_p is the complement of an open set of measure less than 2^{-p}), it follows from Theorem 3 that a bounded measurable function $\psi(t)$ and a decreasing sequence $h_1, h_2, \cdots$ converging to 0 exist such that

$$\limsup_{n\to\infty} | \psi(t + h_n) - \psi(t) | > 0 \tag{19}$$

for almost all t.

4. An application of Theorem 1. In this section we use Theorem 1 to prove the following theorem.

THEOREM 4. *If $x(t)$ and $y(t)$ are measurable over $-\infty < t < \infty$, then the function $G(\lambda)$ defined by*

$$G(\lambda) = \underset{-\infty<t<\infty}{\text{m.u.b.}} \; | x(t + \lambda) - y(t) | \tag{20}$$

is a lower semi-continuous function of λ.

For example, if $x(t) = y(t) = t$, then $G(\lambda) = | \lambda |$. If $x(t) = y(t) = t - [t]$, $[t]$ being the greatest integer $\leqq t$, then $G(\lambda)$ has period 1, $G(0) = 0$ and $G(\lambda) = 1 - | \lambda |$ when $0 < | \lambda | \leqq \frac{1}{2}$. If $x(t) = y(t) = \tan 2\pi t$, then $G(\lambda) = 0$ when $\lambda = 0, \pm 1, \pm 2, \cdots$ and $G(\lambda) = +\infty$ otherwise.

To prove that $G(\lambda)$ is lower semi continuous, assume to the contrary that some λ_0 exists such that

(21) $$\liminf_{\lambda\to\lambda_0} G(\lambda) < G(\lambda_0).$$

Then $\gamma < G(\lambda_0)$ exists such that

(22) $$\liminf_{h\to 0} G(\lambda_0 + h) < \gamma.$$

By Theorem 1, there is a sequence $H_1, H_2, \cdots$ of positive numbers converging to 0 such that if $|h_n| < H_n$, then $\lim x(t + h_n) = x(t)$ for almost all t. The inequality (22) implies existence of a sequence $h_1, h_2, \cdots$ such that $|h_n| < H_n$ and

(23) $$G(\lambda_0 + h_n) < \gamma \qquad n = 1, 2, \cdots.$$

Then, for each $n = 1, 2, \cdots$,

(24) $$|x(t + \lambda_0 + h_n) - y(t)| < \gamma$$

for almost all t. Hence there is a set A_1, whose complement is a null set, such that (24) holds for each $n = 1, 2, \cdots$ when $t \in A_1$. The definitions of H_n and h_n imply existence of a set A_2, whose complement is a null set, such that

(25) $$\lim_{n\to\infty} x(t + \lambda_0 + h_n) = x(t + \lambda_0) \qquad t \in A_2.$$

For each $t \in A_1A_2$ we can let $n \to \infty$ in (24) to obtain

(26) $$|x(t + \lambda_0) - y(t)| \leqq \gamma \qquad t \in A_1A_2.$$

Since the complement of A_1A_2 is a null set, this implies that $G(\lambda_0) \leqq \gamma$. This contradicts the inequality $\gamma < G(\lambda_0)$ and completes the proof of Theorem 4.

5. Periodicity of measurable functions. A function $x(t)$ is periodic, with period h, if $h \neq 0$ and $x(t + h) - x(t) = 0$ for each t in $-\infty < t < \infty$. The following is a classic theorem. If $x(t)$ is continuous, periodic, and not a constant, then $x(t)$ has a least positive period h and each period is an integer multiple of h. That this theorem would fail if the hypothesis of continuity of $x(t)$ were deleted is commonly shown by some such example as the following. Let $x(t) = 1$ when t is rational and $x(t) = 0$ otherwise. Then each positive rational number is a period of $x(t)$, and there is no least one. The function $x(t)$ of this example is *essentially constant*, i.e., there is a constant c such that $x(t) = c$ for almost all t.

Let $x(t)$ be termed *essentially periodic*, with *essential period* h, if $h \neq 0$ and $x(t + h) - x(t) = 0$ for almost all t. Thus $h \neq 0$ is an essential period of $x(t)$ if and only if $F(h) = 0$ where $F(\lambda)$ is defined by

(27) $$F(\lambda) = \underset{-\infty<t<\infty}{\text{m.u.b.}} |x(t + \lambda) - x(t)|.$$

It is easy to show that $F(\lambda)$ is an even function of λ; in particular if h is an essential period of $x(t)$, so also is $-h$.

If a function is periodic, then it is also essentially periodic. Hence all measurable periodic functions which are not essentially constant satisfy the hypothesis of the following theorem.

THEOREM 5. *If $x(t)$ is measurable, essentially periodic, and not essentially constant, then $x(t)$ has a least positive essential period h, and each essential period is an integer multiple of h.*

The hypothesis that $x(t)$ is essentially periodic implies that $x(t)$ has a positive essential period. Hence if $x(t)$ has no least positive essential period, then there is a decreasing sequence $h_1 > h_2 > \cdots$ of positive essential periods. The inequality

$$(28) \quad | x(t + \lambda'') - x(t \pm \lambda') | \leqq | x(t + \lambda'') - x(t) | + | x(t \pm \lambda') - x(t) |$$

and the equality

$$(29) \quad \underset{-\infty<t<\infty}{\text{m.u.b.}} | x(t + \lambda'' \mp \lambda') - x(t) | = \underset{-\infty<t<\infty}{\text{m.u.b.}} | x(t + \lambda'') - x(t \pm \lambda') |$$

imply that, $F(\lambda)$ being defined by (27),

$$(30) \qquad F(\lambda'' \mp \lambda') \leqq F(\lambda'') + F(\lambda').$$

Let $\lambda_n = h_n - h_{n+1}$. Then $\lambda_n > 0$, $\lim \lambda_n = 0$; and using (30) and the fact that $F(h_n) = 0$ for each n, we obtain $F(\lambda_n) = 0$ for each n. Therefore

$$(31) \qquad F(m\lambda_n) = 0 \qquad n = 1, 2, \cdots; m = \pm 1, \pm 2, \cdots.$$

Since Theorem 4 implies that $F(\lambda)$ is lower semi-continuous, and (31) together with the fact that $\lambda_n \to 0$ implies that the set of λ for which $F(\lambda) = 0$ is dense in $-\infty < \lambda < \infty$, it follows that $F(\lambda) = 0$ over $-\infty < \lambda < \infty$.

Let $x(t) = x_1(t) + ix_2(t)$ where $x_1(t)$ and $x_2(t)$ are real. The hypothesis that $x(t)$ is not essentially constant implies that at least one of $x_1(t)$ and $x_2(t)$, say $x_3(t)$, is not essentially constant. Then constants γ_1 and γ_2 exist such that m.l.b. $x_3(t) < \gamma_1 < \gamma_2 <$ m.u.b. $x_3(t)$. Let A and B be bounded sets of positive measure such that $x_3(t) < \gamma_1$ for $t \in A$ and $x_3(t) > \gamma_2$ for $t \in B$. If $\varphi(t)$ and $\psi(t)$ are respectively the characteristic functions of A and B, then

$$(32) \qquad \mu(\lambda) = \int_{-\infty}^{\infty} \varphi(t)\psi(t + \lambda)\, dt$$

is the measure $| C(\lambda) |$ of the set $C(\lambda)$ of points t such that $t \in A$ and $t + \lambda \in B$. Since

$$(33) \qquad \int_{-\infty}^{\infty} \mu(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \varphi(t)\, dt \int_{-\infty}^{\infty} \psi(t + \lambda)\, dt = | A | | B | > 0,$$

we can choose λ_0 such that $\mu(\lambda_0) > 0$. Since

$$(34) \qquad | x_3(t + \lambda_0) - x_3(t) | \geqq \gamma_2 - \gamma_1 > 0 \qquad t \in C(\lambda_0)$$

and $| C(\lambda_0) | = \mu(\lambda_0) > 0$, it follows that

$$(35) \qquad \underset{-\infty<t<\infty}{\text{m.u.b.}} | x_3(t + \lambda_0) - x_3(t) | > 0.$$

Therefore, since x_3 is either x_1 or x_2, $F(\lambda_0) > 0$. This contradicts the previous conclusion that $F(\lambda) = 0$ over $-\infty < t < \infty$, and accordingly proves that $x(t)$ has a least positive period h. Use of Theorem 4 can be avoided by varying the above argument as follows: That fact that

$$(36) \qquad | \mu(\lambda + \delta) - \mu(\lambda) | \leqq \int_{-\infty}^{\infty} | \psi(t + \delta) - \psi(t) | \, dt$$

and the integral on the right converges to 0 as $\delta \to 0$ implies that $\mu(\lambda)$ is continuous. Hence $\mu(\lambda) > 0$ for each λ in some interval containing λ_0 and hence $\mu(\lambda) > 0$ for some λ of the form $m\lambda_n$; this gives a contradiction of (31) without use of Theorem 4.

Existence of a least positive essential period h of $x(t)$ having been established, it follows easily from (30) that each of the numbers $\pm h, \pm 2h, \pm 3h, \cdots$ is an essential period of $x(t)$, and that $x(t)$ can have no other essential period since otherwise the difference of two essential periods would be a positive essential period less than h. This completes the proof of Theorem 5.

If $x(t)$ is periodic, then each period of $x(t)$ is also an essential period of $x(t)$. If $x(t)$ is not essentially constant and h_1 is a period of $x(t)$, then one of the finite set of essential periods of $x(t)$ which are positive and less than or equal to h_1 must be the least positive period of $x(t)$. It is easy to complete the proof of the following theorem.

THEOREM 6. *If $x(t)$ is measurable, periodic, and not essentially constant, then $x(t)$ has a least positive period h and each period is an integer multiple of h.*

It is obvious that if $x(t)$ is essentially constant, then the function

$$(37) \qquad F(\lambda) = \underset{-\infty<t<\infty}{\text{m.u.b.}} | x(t + \lambda) - x(t) |$$

vanishes for $-\infty < \lambda < \infty$, and Theorem 5 implies the converse. This fact and Theorem 5 imply that if $x(t)$ is measurable, then the set of Z of zeros of $F(\lambda)$, which must be closed by Theorem 4, must have one of three simple forms. Either $F(\lambda) \neq 0$ for $\lambda \neq 0$, in which case $x(t)$ is not essentially periodic; or $h > 0$ exists such that $F(\lambda) = 0$ when $\lambda = 0, \pm h, \pm 2h, \cdots$ and $F(\lambda) \neq 0$ otherwise, in which case $x(t)$ is essentially periodic but not essentially constant; or $F(\lambda) = 0$ over $-\infty < t < \infty$ in which case $x(t)$ is essentially constant. Likewise the set of zeros of $G(\lambda)$, defined by (20) must be very simple; for if $G(\lambda) = 0$ for some λ, say λ_0, then $y(t) = x(t - \lambda_0)$ for almost all t and therefore $G(\lambda) = F(\lambda - \lambda_0)$.

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THE DIMENSION OF THE RATIONAL POINTS IN HILBERT SPACE

BY PAUL ERDÖS

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Let H denote the Hilbert-space consisting of all sequences of real numbers

$$\xi = (x_1, x_2, \cdots) \tag{1}$$

such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty$$

with the distance defined as usual. R will denote the set of points of H having all coordinates rational. R_0 will denote the set of points of H of the form

$$\nu = \left(\frac{1}{n_1}, \frac{1}{n_2}, \cdots\right) \tag{2}$$

where n_i are positive integers.

Let $R_1 = \bar{R}_0$. Clearly $R_0 \subset R_1 \subset R$.

THEOREM.[1] Dim R_0 = dim R_1 = dim R = 1.

Before we proceed with the proof let us remark that the cartesian product $R_1 \times R_1$ is homeomorphic to R_1. Hence we obtain that

There exists a metric separable complete space X such that dim X = dim $X \times X = 1$.

This seems to be a new contribution to the "product problem"[2] of the theory of dimensions. It might also be worth noticing that R_1 is disconnected between any two of its points.

Proof that dim $R_0 > 0$. Let U be an open subset of H of diameter less than $\frac{1}{2}$ and such that $UR_0 \neq 0$. Let therefore (2) belong to U.

We shall define a sequence of natural numbers $m_1, m_2, \cdots$ such that

$$\nu_k = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_{k-1}}, \frac{1}{m_k}, \frac{1}{n_{k+1}}, \frac{1}{n_{k+2}}, \cdots\right) \in U \tag{3}$$

$$\mu_k = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_{k-1}}, \frac{1}{m_k - 1}, \frac{1}{n_{k+1}}, \frac{1}{n_{k+2}}, \cdots\right) \notin U. \tag{4}$$

Suppose that the m_i are already defined for $i < k$. Let m_k be the smallest integer such that (3) holds. Since the diameter of U is less than $\frac{1}{2}$ it is clear that $m_k > 1$ and that (4) holds.

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[1] The problem of computing the dimension of R was proposed to me by Professor W. Hurewicz.

[2] See L. Pontrjagin, C.R. Paris 190 (1930), p. 1105 and W. Hurewicz, Ann. of Math. 36 (1935), p. 194.

Since U is bounded there is a number N such that $|\nu_k| < N$. Therefore $\sum_{i=1}^{\infty}\left(\frac{1}{m_i}\right)^2 < \infty$ and the point

$$\mu = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots\right)$$

is in R. Clearly $\mu = \lim \nu_k$. Since $|\nu_k - \mu_k| = \dfrac{1}{m_k(m_k - 1)}$ and $m_k \to \infty$ we have also $\mu = \lim \mu_k$. In view of (3) and (4) μ is then on the boundary of U.

This proves that R_0 has positive dimension at every one of its points. The same holds for R_1 and R.

Proof that dim $R \leqq 1$. Let S be the sphere consisting of all points (1) such that

$$\sum_{i=1}^{\infty} x_i^2 = 1.$$

It is clearly sufficient to prove that dim $R \cdot S = 0$. Let

$$\rho = (r_1, r_2, \cdots)$$

be any point of $R \cdot S$. Given any positive irrational number δ choose n so that

$$\sum_{i=n+1}^{\infty} r_i^2 < \delta. \tag{5}$$

Let V_δ be the set of all points (1) such that

$$\sum_{i=1}^{n} r_i x_i > 1 - \delta.$$

Clearly V_δ is an open set. If a point (1) is on the boundary of V_δ then

$$\sum_{i=1}^{n} r_i x_i = 1 - \delta$$

hence $x_1, x_2, \cdots, x_n$ cannot all be rational. We have proved therefore that the boundary of V_δ contains no point of R.

Since $\rho \in S$ we have $\sum_{i=1}^{\infty} r_i^2 = 1$, therefore by (5) $\sum_{i=1}^{n} r_i^2 > 1 - \delta$ and $\rho \in V_\delta$.

To finish the proof it is therefore sufficient to prove that the diameter of $S \cdot V_\delta$ tends to zero as δ tends to zero. Let

$$\xi = (x_1, x_2, \cdots)$$

be a point if $S \cdot V_\delta$. We have then

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 = \sum_{i=1}^{\infty} r_i^2 + \sum_{i=1}^{\infty} x_i^2 - 2\sum_{i=1}^{n} r_i x_i - 2\sum_{i=n+1}^{\infty} r_i x_i.$$

Since ρ and ξ are in S and ξ is in V_δ therefore

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 < 1 + 1 - 2(1 - \delta) - 2 \sum_{i=n+1}^{\infty} r_i x_i.$$

Using Schwarz's inequality and (5) we have

$$\sum_{i=n+1}^{\infty} r_i x_i \leqq \left(\sum_{i=1}^{\infty} r_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}} < \delta^{\frac{1}{2}}.$$

Therefore

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 < 2\delta + 2\delta^{\frac{1}{2}}$$

which completes the proof.

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ON TWO FORMULATIONS OF THE THEORY OF STOCHASTIC PROCESSES DEPENDING UPON A CONTINUOUS PARAMETER

By J. L. Doob and W. Ambrose

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Introduction

The theory of stochastic processes depending upon a continuous parameter is the theory of measure (probability) relations on a collection of functions, $\{x(t)\}$, t ranging over the real numbers. There is some difficulty however in finding an appropriate collection of functions on which to consider the measure relations. On the one hand it is desirable to consider as large a class of functions as possible while on the other hand it is desirable that the functions considered have enough regularity properties, both individually and as a class, that one may systematically make use of known theorems from the theory of functions in investigating these measure relations. One way of choosing the collection of functions to be considered has been given by Doob (II);[1] he considers first a measure defined on the space of all real-valued functions, $x(t)$, and carries this measure over to certain subspaces. Then he shows that in certain cases these subspaces will have desirable regularity properties. Another approach was given by Wiener (V and VIII), who takes a function, $f(t, x)$, subject to certain regularity conditions, and then considers the collection of t-functions obtained from $f(t, x)$ by fixing x and allowing t to vary. Wiener defines a measure on this space of t-functions in terms of a measure on x-space. The principal result of the present paper is the establishing of some relations between these two approaches to the theory of stochastic processes depending upon a continuous parameter. In section 1 we give the precise formulations of these two kinds of stochastic processes, in section 2 we show their equivalence, and in section 3 we obtain some further theorems relating the two.

1. Definitions

In this section we shall give the formulations of Doob and of Wiener of the notion of a stochastic process depending upon a continuous parameter. We shall then establish an equivalence relation between these two formulations.

Let Ω^* be the space of all real-valued functions of a real variable. We introduce a topology on Ω^* by defining neighborhoods as follows: if $t_1, \cdots, t_n$ is any finite set of real numbers and if $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are real numbers

[1] Roman numerals refer to the bibliography at the end of this paper.

satisfying $-\infty \leqq a_i < b_i \leqq \infty$ $(i = 1, \cdots, n)$ then the set of elements, $x(t)$, of Ω^* which satisfy

$$a_i < x(t_i) < b_i, \qquad i = 1, \cdots, n,$$

is a neighborhood. Let $P^*(\Lambda^*)$ be a probability measure[2] which is defined for those sets in Ω^* which are Borel sets, and then complete this measure.[3] Those sets for which P^*-measure then is defined will be called P^*-measurable. We define the outer P^*-measure for every set, M^*, to be the lower bound of numbers of the form $P^*(\Lambda^*)$, where Λ^* ranges over the P^*-measurable sets which include M^*.

Definition 1. Let Ω be a subset of Ω^* of outer P^*-measure 1. If Λ is any set of the form $\Lambda = \Lambda^*\Omega$, where Λ^* is P^*-measurable, then we define the probability measure, $P(\Lambda)$, by $P(\Lambda) = P^*(\Lambda^*)$.[4] The space Ω together with the probability measure $P(\Lambda)$ is a *stochastic process* (as defined by Doob in II).

Those sets, Λ, for which $P(\Lambda)$ is defined will be called P-measurable. Although a stochastic process as defined here is a space plus a measure, we shall frequently find it convenient to speak as though the space alone were the stochastic process (as when we speak of Ω^* containing a stochastic process). The context will always make the meaning clear.

We shall now give Wiener's definition of a stochastic process. Let X be any space of points, x, on which a probability measure, $m(E)$, is defined.[5]

Definition 2. A *random function* is a function, $f(t, x)$, defined on $T \times X$ (where T is the space of real numbers and X is a space with the properties just mentioned) which is a measurable function of x for each value of t.

In this formulation of stochastic processes the space of t-functions on which the measure is considered is the collection of t-functions that can be obtained from $f(t, x)$ (where $f(t, x)$ is a random function) by fixing x and allowing t to vary. Probability relations on this space of t-functions are defined in terms of the probability measure which is given on x-space. If t and k are real numbers, then the probability that $x(t_0) < k$ (or, in other words, the measure of the set of t-functions, $x(t)$, for which $x(t_0) < k$) is defined to be

$$\underset{x}{m}[f(t_0, x) < k].$$

Then in terms of these elementary probabilities it is possible to define more complicated probabilities on this space of t-functions; i.e., having this measure defined by a random function for every collection of t-functions, $x(t)$, which is defined by a condition of the form: $x(t_0) < k$, it is possible to extend this measure

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[2] A probability measure is a completely additive non-negative set function for which $P^*(\Omega^*) = 1$.

[3] A measure is completed if it is defined for every subset of each set whose measure is zero; it is possible to complete any measure.

[4] It is shown in II that this uniquely defines $P(\Lambda)$. It is easy to see that $P(\Lambda)$ is a completed measure.

[5] Strictly speaking, the measure is not defined on X but rather on those subsets of X which lie in a certain Borel field.

to the Borel field[6] determined by these collections of t-functions. This space of t-functions together with the probability measure thus defined is a stochastic process in the sense of Wiener.[7]

2. The Equivalence of the Two Kinds of Stochastic Processes

We shall now establish the equivalence of the two kinds of stochastic processes whose formulations we have just given. Theorems 1 and 2 of this section together show that a stochastic process which is determined by a random function is a stochastic process in the sense of Definition 1. Theorem 1 shows that with a proper P^*-measure on Ω^* the collection of t-functions obtained from $f(t, x)$ by fixing x and letting t vary will have outer P^*-measure 1; then Theorem 2 shows that the P-measure on this collection of functions will be the same as the measure on them which is obtained from x-measure by the method of Wiener. A few remarks suffice to show, conversely, that a stochastic process in the sense of Definition 1 is always determined by a random function.

Let $f(t, x)$ be a random function; if Λ^* is any set in Ω^* then we denote by

$$\underset{x}{[f(t, x) \in \Lambda^*]}$$

the collection of x-points for which the t-function $f(t, x)$ belongs to Λ^*. We now define a P^*-measure on Ω^* in terms of $f(t, x)$; if Λ^* is any Borel set in Ω^* then its P^*-measure is defined by

$$P^*(\Lambda^*) = m\underset{x}{[f(t, x) \in \Lambda^*]}.^{8}$$

Upon completing this P^*-measure we have a P^*-measure of the type considered in section 1. We shall refer to this P^*-measure as the P^*-measure *determined by* $f(t, x)$. We shall denote by $\Omega(f)$ the collection of t-functions obtained from $f(t, x)$ by fixing x and allowing t to vary.

THEOREM 1. *If $f(t, x)$ is a random function and if P^*-measure is the measure on Ω^* determined by $f(t, x)$ then $\Omega(f)$ is a stochastic process (in the sense of Definition 1).*

PROOF: To prove this theorem we must show that the outer P^*-measure of $\Omega(f)$ is equal to 1; to prove this it is sufficient to show that if Λ^* is any Borel set in Ω^* which includes $\Omega(f)$ then $P^*(\Lambda^*) = 1$. Let, then, Λ^* be any Borel set in Ω^* which includes $\Omega(f)$; the fact that Λ^* includes $\Omega(f)$ implies that $X = \underset{x}{[f(t, x) \in \Lambda^*]}$. Using this and the definition of P^*-measure we have,

$$P^*(\Lambda^*) = m\underset{x}{[f(t, x) \in \Lambda^*]} = m(X) = 1,$$

as was to be proved.

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[6] This Borel field is considered more carefully later on.

[7] This formulation of a stochastic process is more general than that given in V. We shall also refer to it as a stochastic process determined by a random function, $f(t, x)$.

[8] This definition of course assumes the easily verified fact that if Λ^* is any Borel set in Ω^* then $\underset{x}{[f(t, x) \in \Lambda^*]}$ is a measurable x-set.

If $f(t, x)$ is a random function then with every pair of numbers, t and k, we can associate the x-set,

$$\underset{x}{[f(t, x) < k]},$$

and we shall denote by $\mathcal{F}$ the Borel field of x-sets determined by x-sets of this form; because every set in $\mathcal{F}$ will be x-measurable we have a probability measure (the original measure on X) defined for sets of $\mathcal{F}$. We denote by $\overline{\mathcal{F}}$ the collection of x-sets obtained by completing $\mathcal{F}$ with respect to this measure.

Now consider the correspondence, $x \rightarrow \omega$, of X into $\Omega(f)$ defined by making correspond to each x the t-function, $f(t, x)$. If Λ is any set in $\Omega(f)$ then we denote its origin[9] under this correspondence by Λ', i.e., $\Lambda' = \underset{x}{[f(t, x) \epsilon \Lambda]}$. If Λ^* is any set in Ω^* and if $\Lambda = \Lambda^*\Omega$ then it is obvious that

$$\underset{x}{[f(t, x) \epsilon \Lambda]} = \underset{x}{[f(t, x) \epsilon \Lambda^*]}.$$

It is easy to see that although the transformation $x \rightarrow \omega$ does not necessarily have a single-valued inverse, every x-set Λ' in $\mathcal{F}$ is the origin of an ω-set Λ. The following theorem describes the exact relation between P-measure and x-measure.

THEOREM 2. *Λ is P-measurable if and only if $\Lambda' \epsilon \overline{\mathcal{F}}$; also $m(\Lambda') = P(\Lambda)$.*[10]

PROOF: We shall show first that if $\Lambda = \Lambda^*\Omega(f)$ where Λ^* is a Borel set in Ω^* then $\Lambda' \epsilon \mathcal{F}$. This is obvious in case Λ^* is a neighborhood and it is easily verified that the collection of sets, Λ^*, for which $\Lambda' \epsilon \mathcal{F}$ is a Borel field; because the Borel sets constitute the smallest Borel field in Ω^* which includes all neighborhoods it follows that if Λ^* is a Borel set then Λ' will be in $\mathcal{F}$. By a similar proof one can show that if $\Lambda' \epsilon \mathcal{F}$ and Λ' is the origin of Λ then Λ must be the intersection of $\Omega(f)$ with a Borel set in Ω^*. In this case, where $\Lambda' \epsilon \mathcal{F}$, we see directly from the definitions of $P^*(\Lambda^*)$ and $P(\Lambda)$ that $m(\Lambda') = P(\Lambda)$.

We shall now show that if Λ is any P-measurable set then $\Lambda' \epsilon \overline{\mathcal{F}}$ and $P(\Lambda) = m(\Lambda')$. If Λ is any P-measurable set then $\Lambda = \Lambda_1 + \Lambda_2$ where Λ_1 is of the form $\Lambda_1^*\Omega(f)$, with Λ_1^* a Borel set in Ω^*, and $\Lambda_2 = \Lambda_2^*\Omega(f)$, with Λ_2^* a subset of a Borel set, Λ_3^*, whose P^*-measure is zero. Then $\Lambda' = \Lambda_1' + \Lambda_2'$ where $\Lambda_1' \epsilon \mathcal{F}$ and $m(\Lambda_1') = P(\Lambda_1)$ (this was proved in the preceeding paragraph). Λ_2' is a subset of Λ_3' for which (again by the preceeding paragraph) $m(\Lambda_3') = 0$. Therefore $m(\Lambda_2') = 0$; hence $\Lambda' \epsilon \overline{\mathcal{F}}$ and $m(\Lambda') = P(\Lambda)$. Similarly it can be shown that if $\Lambda' \epsilon \overline{\mathcal{F}}$ then Λ is P-measurable. This completes the proof.

Having shown in detail how a random function determines a stochastic process (i.e., a stochastic process in the sense of Definition 1), we shall now show,

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[9] We use the word "origin" rather than the word "inverse" because this correspondence need not be one to one.

[10] In the statement and proof of this theorem we adhere to a fixed notation with Λ^* for a set in Ω^*, Λ the intersection of Λ^* with $\Omega(f)$, and Λ' the origin of Λ under the correspondence between X and $\Omega(\mathcal{F})$.

conversely, that any stochastic process in the sense of Definition 1 is determined by a random function. Let Ω be such a stochastic process and consider the function $x_t(\omega)$, whose domain is $T \times \Omega$ (where T is the space of real numbers), defined as follows: if (t_0, ω_0) is any point in $T \times \Omega$ then the value of $x_t(\omega)$ at that point is the value which the function $\omega_0(\omega_0 \in \Omega^*$ and hence is a function of a real variable) takes when $t = t_0$. Then $x_t(\omega)$ will be a random function which determines the stochastic process.

3. Random Functions and Measurable Stochastic Processes

In the formulation of the theory of stochastic processes a fundamental notion is that of a *measurable* stochastic process.[11] In this section we first give the definition of a measurable stochastic process and then we characterize (Theorem 3) those random functions from which the stochastic process obtained will be measurable in this sense. We conclude with some further theorems about such functions and about measurable stochastic processes.

If Ω is any stochastic process, then in terms of P-measure on Ω and Lebesgue measure on T we can define a measure on the product space $T \times \Omega$ in the usual (multiplicative) way; we suppose also that this measure on $T \times \Omega$ is completed.

DEFINITION 3. If Ω is a stochastic process such that $x_t(\omega)$ is measurable on $T \times \Omega$ then Ω is a *measurable* stochastic process.[12]

Let $f(t, x)$ be a random function and let $\overline{\mathfrak{F}}$ be the Borel field of x-sets associated with it, as defined in section 1. Let $\overline{\mathfrak{T}}$ denote the collection of Lebesgue measurable sets on the t-axis with Lebesgue measure defined on them. We denote by $\overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$ the smallest Borel field of sets in $T \times X$ which contains all sets of the form $E \times F$, where $E \in \overline{\mathfrak{T}}$ and $F \in \overline{\mathfrak{F}}$, and we denote by $\overline{\mathfrak{T} \times \mathfrak{F}}$ the collection of sets obtained by completing $\overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$ with respect to the usual (multiplicative) measure defined on it.

DEFINITION 4. $f(t, x)$ is *strongly measurable* if it is measurable with respect to $\overline{\mathfrak{T} \times \mathfrak{F}}$.[13]

In section 1 we considered a correspondence, determined by a random function, $f(t, x)$, between X and $\Omega(f)$ (where $\Omega(f)$ is the collection of t-functions obtained from $f(t, x)$ by fixing x and allowing t to vary). We consider now the product spaces $T \times X$ and $T \times \Omega(f)$ and in terms of the correspondence between X and $\Omega(f)$ we define a correspondence between these product spaces as follows: if $x_0 \rightarrow \omega_0$ under the correspondence between X and $\Omega(f)$, then we define the new correspondence by making $(t, x_0) \rightarrow (t, \omega_0)$ for every value of t. If $\tilde{\Lambda}$ is a set

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[11] See II where such processes are discussed in detail.

[12] That this definition is non-trivial is shown in II, Theorem 2.2, p. 113. The function $x_t(\omega)$ was defined in the last section.

[13] Because $\overline{\mathfrak{F}}$ is, in general, a subcollection of the x-measurable sets, the condition of strong measurability is more stringent than the condition of (t, x)-measurability. Theorem 5 below shows that if $f(t, x)$ is a measurable (t, x)-function, it will be measurable with respect to some field $\overline{\mathfrak{T} \times \mathfrak{F}_1}$ where $\mathfrak{F}_1$ contains besides the sets of $\mathfrak{F}$, certain sets of x-measure 0.

in $T \times \Omega(f)$ we shall denote its origin under this correspondence by $\tilde{\Lambda}'$. Using this notation we have

LEMMA. *$\tilde{\Lambda}$ is measurable in $T \times \Omega(f)$ if and only if $\tilde{\Lambda}' \epsilon \overline{\mathfrak{T} \times \mathfrak{F}}$. If $\tilde{\Lambda}$ is measurable in $T \times \Omega(f)$ then its measure is equal to the measure of $\tilde{\Lambda}'$.*

PROOF: Let $\tilde{\mathfrak{G}}$ be the Borel field of sets in $T \times \Omega(f)$ determined by the sets which are direct products of sets in $\overline{\mathfrak{T}}$ and P-measurable sets so that $\tilde{\mathfrak{G}}$ differs from the collection of measurable sets in $T \times \Omega(f)$ in that $\tilde{\mathfrak{G}}$ is not completed with respect to the measure which is defined on it. Then, as in the proof of Theorem 2, we can show that $\tilde{\Lambda} \epsilon \tilde{\mathfrak{G}}$, if and only if $\tilde{\Lambda}' \epsilon \overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$. Denoting by Λ_t and by Λ'_t respectively the ω-set for which $(t, \omega) \epsilon \tilde{\Lambda}$ and the x-set for which $(t, x) \epsilon \tilde{\Lambda}'$ we see that if $\tilde{\Lambda}'$ is the origin of $\tilde{\Lambda}$ under the correspondence on the product spaces, then for each t, Λ'_t is the origin of Λ_t under the correspondence between X and $\Omega(f)$. If $\tilde{\Lambda}' \epsilon \overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$ each $\Lambda'_t \epsilon \overline{\mathfrak{F}}$, and if $\tilde{\Lambda} \epsilon \tilde{\mathfrak{G}}$, each Λ_t is P-measurable. Hence, by Theorem 2, if $\tilde{\Lambda}' \epsilon \overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$

$$P(\Lambda_t) = m(\Lambda'_t),$$

for all t, which (by Fubini's theorem) implies that in this case $\tilde{\Lambda}$ and $\tilde{\Lambda}'$ have the same measure.

Now, knowing this theorem to be true if we consider only the Borel fields $\tilde{\mathfrak{G}}$ and $\overline{\mathfrak{T}} \times \overline{\mathfrak{F}}$ an argument like that used in the last part of the proof of Theorem 2 completes the proof of this lemma.

THEOREM 3. *If $f(t, x)$ is a random function then $\Omega(f)$ is a measurable stochastic process if and only if $f(t, x)$ is strongly measurable.*

PROOF: Let k be a real number and consider the sets:

$$\tilde{\Lambda}(k) = \underset{(t,\omega)}{[x_t(\omega) > k]} \qquad \tilde{\Lambda}'(k) = \underset{(t,x)}{[f(t, x) > k]}.$$

It is easily seen that (for every value of k) $\tilde{\Lambda}'(k)$ is the origin of $\tilde{\Lambda}(k)$ under the correspondence considered above. By the Lemma just proved we see that $\tilde{\Lambda}(k)$ is measurable in $T \times \Omega(f)$ for every value of k if and only if $\tilde{\Lambda}'(k) \epsilon \overline{\mathfrak{T} \times \mathfrak{F}}$ for every value of k, i.e., that $x_t(\omega)$ is measurable on $T \times \Omega(f)$ if and only if $f(t, x)$ is strongly measurable.

Theorem 3 makes it desirable to have some explicit conditions under which a random function will be strongly measurable. For that reason we state a theorem which has been proved by Ursell.[14]

THEOREM 4. *If $f(t, x)$ is a random function which satisfies either one of the following conditions, then it is strongly measurable*:

1) *for each value of x the t-function $f(t, x)$ is continuous.*

2) *for each value of x the t-function $f(t, x)$ is monotonic.*

Because of Theorem 3, we see that a random function $f(t, x)$ may be (t, x)-measurable without $\Omega(f)$ necessarily being a measurable stochastic process.

[14] See Theorems 8 and 13, pp. 326–330, of VII. The theorems stated by Ursell are not as strong as the theorem which we state; with trivial modifications, however, his proofs can be used to establish our theorem. The authors are indebted to Dr. P. R. Halmos for some very helpful discussions of these matters.

We now consider a different question: if $f(t, x)$ is a random function which is (t, x)-measurable, and if P^*-measure is taken to be the measure determined by $f(t, x)$ (see section 1), then must Ω^* contain a measurable stochastic process? The answer, as given by Theorem 6, is in the affirmative.

DEFINITION 5. If $f(t, x)$ is a random function and if $g(t, x)$ is a function which, for each t, equals $f(t, x)$ except for an x-set (depending on t) of measure zero, then $g(t, x)$ is *equivalent to* $f(t, x)$.

THEOREM 5. *If $f(t, x)$ is a random function, then a necessary and sufficient condition that there exist an equivalent function which is strongly measurable is that there exist an equivalent function which is (t, x)-measurable.*

PROOF: Because the necessity of the condition is obvious, we need prove only its sufficiency. Let $g(t, x)$ be a (t, x)-measurable function which is equivalent to the random function $f(t, x)$. Define $\varphi_n(t)$ by

$$\varphi_n(t) = k2^{-n} \text{ if } (k-1)2^{-n} < t \leqq k2^{-n} \qquad \begin{matrix} k = 0, \pm 1, \cdots, \\ n = 1, 2, \cdots. \end{matrix}$$

Let c be any real number. Since for $(k-1)2^{-n} < t \leqq k2^{-n}$

$$g[\varphi_n(t) + c, x] = g[k2^{-n} + c, x],$$

$g[\varphi_n(t) + c, x]$ is (t, x)-measurable on each (t, x)-measurable set determined by $(k-1)2^{-n} < t \leqq k2^{-n}$, so that $g[\varphi_n(t) + c, x]$ is (t, x)-measurable. It has been shown that there is a number c, and a sequence $\{a_n\}$ of integers such that

$$\lim_{n\to\infty} g[\varphi_{a_n}(t) + c, x] = g(t + c, x)$$

for almost all (t, x).[15] This implies that there is a t-set S of Lebesgue measure 0 such that if $t \notin S$,

$$\lim_{n\to\infty} g[\varphi_{a_n}(t) + c, x] = g(t + c, x)$$

for almost all x. Then if $t \notin S$,

$$\lim_{n\to\infty} f[\varphi_{a_n}(t) + c, x] = f(t + c, x)$$

for almost all x. Since

$$\varphi_\nu(k2^{-n}) = k2^{-n}, \qquad \nu \geqq n,$$

$$g[\varphi_{a_j}(k2^{-n}) + c, x] = g[k2^{-n} + c, x], \qquad a_j \geqq n,$$

so that S does not include any $k2^{-n} + c$. Define $f_1(t, x)$ by $f_1(t + c, x) = f(t, x)$, $t \notin S$

$$f_1(t + c, x) = \limsup_{n\to\infty} f[\varphi_{a_n}(t) + c, x], \qquad t \in S,$$

[15] Doob II, p. 113. The proof is given there for x-space the interval $0 < x < 1$, but the proof is perfectly general.

if the superior limit is finite. The function $f[\varphi_n(t) + c, x]$ is (t, x)-measurable, by the same proof used for $g[\varphi_n(t) + c, x]$. Then the above limit superior is $\pm\infty$ on at most a (t, x)-measurable set of measure 0, and we define $f_1(t, x)$ as 0 on this set. The function $f_1(t, x)$ is a (t, x)-measurable random function, equivalent to $f(t, x)$, and

$$f_1(k2^{-n} + c, x) = f(k2^{-n} + c, x) \qquad \begin{array}{l} k = 0, \pm 1, \cdots, \\ n = 1, 2, \cdots . \end{array}$$

We shall show that $f_1(t, x)$ is strongly measurable. Let $\mathcal{F}_1$ be the smallest Borel field of x-sets including the x-sets

$$\underset{x}{[}f_1(k2^{-n} + c, x) < l] = \underset{x}{[}f(k2^{-n} + c, x) < l], \qquad \begin{array}{l} k = 0, \pm 1, \cdots, \\ n = 1, 2, \cdots, \\ -\infty < l < \infty . \end{array}$$

It will be sufficient to show that $f_1(t, x)$ is measurable with respect to the field $\overline{\mathcal{T} \times \mathcal{F}_1}$. The proof of the (t, x)-measurability of $f[\varphi_n(t) + c, x]$ shows the measurability of that function with respect to the field $\overline{\mathcal{T} \times \mathcal{F}_1}$. Therefore $\limsup_{n\to\infty} f[\varphi a_n(t) + c, x]$ is also measurable with respect to the field $\overline{\mathcal{T} \times \mathcal{F}_1}$, and $f_1(t, x)$ only differs from this superior limit on a set in $\overline{\mathcal{T} \times \mathcal{F}_1}$ of measure 0, so $f_1(t, x)$ is measurable with respect to the field $\overline{\mathcal{T} \times \mathcal{F}_1}$, as was to be shown.

COROLLARY. *If $f(t, x)$ is a random function, and if there is a (t, x)-measurable function $g(t, x)$ such that when t is not in some set, E, of Lebesgue measure 0, $f(t, x) = g(t, x)$ for almost all x, then $f(t, x)$ is equivalent to a strongly measurable function.*

Unless the set E is empty, Theorem 5 is not immediately applicable, but we can redefine $g(t, x)$, for $t \in E$, to be equal to $f(t, x)$, obtaining a (t, x)-measurable function equivalent to $f(t, x)$, and then apply Theorem 5.

We now combine Theorem 5 with Theorem 3 to obtain

THEOREM 6. *Let $f(t, x)$ be a random function, and let P^*-measure be the measure on Ω^* determined by $f(t, x)$.*[16] *Then, if $f(t, x)$ is equivalent to a measurable (t, x)-function, Ω^* will contain a measurable stochastic process.*

PROOF: If $f(t, x)$ is equivalent to a measurable (t, x)-function, then, by Theorem 5, it will be equivalent to a strongly measurable function, $g(t, x)$; this function will of course determine the same P^*-measure as $f(t, x)$. Because $g(t, x)$ is strongly measurable, Theorem 3 implies that the t-functions obtained from it by fixing x and allowing t to vary will form a measurable stochastic process.

This theorem, besides relating the two formulations of a stochastic process which were considered in section 1, is useful because it enables one to prove an important condition, due to Kolmogoroff,[17] for the existence of a measurable

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[16] The P^*-measure determined by $f(t, x)$ is defined in section 1.

[17] See I, p. 71, where the Kolmogoroff theorem has been obtained from a special case of Theorem 6.

stochastic process and because it shows that to find a condition upon P-measure that there be a measurable process with that measure, it is only necessary to find a condition upon a random function $f(t, x)$ that it be equivalent to a measurable (t, x)-function.

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MINIMALLY ALMOST PERIODIC GROUPS

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1. Given a group $\mathfrak{G}$ it is of some interest to decide which elements of $\mathfrak{G}$ can be "told apart" by almost periodic functions of $\mathfrak{G}$ or, which is the same thing (cf. below) by finite dimensional bounded linear representations of $\mathfrak{G}$. That is: *For two* $a, b \in \mathfrak{G}$ *we define* $a \sim b$ *by either of these two properties*:

(I) *For every almost periodic function* $f(x)$ *in* $\mathfrak{G}$ $f(a) = f(b)$.

(II) *For every finite dimensional linear unitary representation* $D(x)$ *of* $\mathfrak{G}$ $D(a) = D(b)$.

The general theory of almost periodic functions in groups[1] deals with these questions. The equivalence of (I) and (II) is shown in Ap, p. 480, Theorem 33. It is also known there, Ap, p. 481, Theorem 34, that the above notion determines an invariant subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$, such that $a \sim b$ holds if and only if a and b belong to the same coset of $\mathfrak{G}_0$ in $\mathfrak{G}$.

The smaller $\mathfrak{G}_0$, the more almost-periodic functions—and finite-dimensional bounded linear representations—$\mathfrak{G}$ possesses. Hence we termed $\mathfrak{G}$ *maximally almost periodic if* $\mathfrak{G}_0 = (1)$ (i.e., $a \sim b$ for $a = b$ only), and *minimally almost periodic* if $\mathfrak{G}_0 = \mathfrak{G}$ (i.e. always $a \sim b$), cf. Ap, p. 482, Def. 17.

Maximally almost-periodic groups $\mathfrak{G}$ are easily constructed; obviously every finite dimensional, linear, unitary group is such. For more general examples cf. Ap, pp. 482–483, Theorem 36, A and B.

Minimally almost-periodic groups $\mathfrak{G}$ are somewhat pathological; they are characterized by the fact that their only almost-periodic functions are the constants: or equivalently, their only finite dimensional, linear, unitary representation is $D(x) \equiv 1$. The existence of such groups was shown in Ap, p. 483, using an interesting theorem of B. L. van der Waerden, combined with earlier results of J. v. Neumann and the representation theory of the affine group. (Cf. loc. cit.)

Thus these examples are really obtained in a rather complicated way.

In this note we will construct examples of various kinds of groups, including minimally almost-periodic ones, by simpler direct methods. All these examples will be enumerably infinite (discrete) groups, whereas those of Ap (cf. above) are continuous (Lie) groups. We will also gain some general viewpoints concerning the nature of maximal and minimal almost periodicity.

2. If $\mathfrak{G}'$ is an invariant subgroup of $\mathfrak{G}$ with finite index, then its factor group $\mathfrak{G}/\mathfrak{G}'$ is finite. Hence the regular representation of $\mathfrak{G}/\mathfrak{G}'$ is finite dimensional,

[1] J. v. Neumann, *Almost periodic functions in a group*, Amer. Math. Soc. Trans. 36 (1934), pp. 445–492. To be quoted as "Ap".

linear, unitary. It is the same for $\mathfrak{G}$. It represents any two different elements of $\mathfrak{G}/\mathfrak{G}'$—hence any two elements of which belong to different cosets of $\mathfrak{G}'$ in $\mathfrak{G}$—differently. Hence $\mathfrak{G}_0 \subseteq \mathfrak{G}'$.

We see therefore: *If $\mathfrak{G}_1$ is the intersection of all invariant subgroups $\mathfrak{G}'$ of $\mathfrak{G}$ with finite index, then*

$$\mathfrak{G}_0 \subseteq \mathfrak{G}_1 .$$

We do not have in general, however, $\mathfrak{G}_0 = \mathfrak{G}_1$. Let $\mathfrak{G}$ be the group of all proper rotations in 3 dimensions. Since $\mathfrak{G}$ is itself a finite dimensional, linear, unitary group, so it is maximally almost periodic. Hence $\mathfrak{G}_0 = (1)$. On the other hand one verifies easily that the only invariant subgroup $\mathfrak{G}'$ of $\mathfrak{G}$ with finite index is $\mathfrak{G}' = \mathfrak{G}$, therefore $\mathfrak{G}_1 = \mathfrak{G}$.

3. Abelian groups—if they are separable and locally compact, which is the case for the enumerably infinite (discrete) ones—are maximally almost periodic, cf. Ap, pp. 482–483, Theorem 36, B. Hence one might be led to expect that the minimally almost-periodic groups—especially the enumerably infinite ones—will be found at the other extreme: among the free groups. This is not so, however. The free group of $h(= 2, 3, \cdots)$ generators, $\mathfrak{G} = \mathfrak{F}^{(h)}$, is maximally almost periodic. Indeed, the reader who is familiar with the general theory of the $\mathfrak{F}^{(h)}$ will find no difficulty in constructing for any $a \in \mathfrak{F}^{(h)}$, $a \neq 1$, an invariant subgroup $\mathfrak{G}'$ of $\mathfrak{F}^{(h)}$ with finite index, such that $a \notin \mathfrak{G}'$. Hence we have $\mathfrak{G}_1 = (1)$ and *a fortiori* $\mathfrak{G}_0 = (1)$ for $\mathfrak{G} = \mathfrak{F}^{(h)}$, cf. §2 above. (For another example of a maximally almost-periodic group which is closely related to free groups, cf. $\mathfrak{G} = \mathfrak{G}^{(\beta)}$ in §5 below.)

Thus both extremes—Abelian groups as well as free groups—are maximally almost periodic. The minimally almost periodic groups must be somewhere in between.

4. Our examples are based on the following lemma:

LEMMA 1. *Let U be a finite dimensional unitary matrix. Assume that there exists for every s $(= 1, 2, \cdots)$ a $t = t(s)$ $(= 1, 2, \cdots)$ which is an integral multiple of s such that for a suitable pair of reciprocal matrices V_t, V_t^{-1} there is $U^t = V_t U V_t^{-1}$. Then $U = 1$.*

PROOF: Let n $(= 1, 2, \cdots)$ be the dimensionality (order) of the matrices under consideration. Let $\alpha_1, \cdots, \alpha_n$ be the characteristic values of U,—in an arbitrary order, but with the right multiplicities.

Consider a $t = t(s)$. U^t has the characteristic values $\alpha_1^t, \cdots, \alpha_n^t$ while $V_t U V_t^{-1}$ has again the characteristic values $\alpha_1, \cdots, \alpha_n$. Therefore the $\alpha_1, \cdots, \alpha_n$ must be a permutation of the $\alpha_1^t, \cdots, \alpha_n^t$.

Since there are infinitely many different $t = t(s)$ (as $t(s) \geqq s$ so $s \to \infty$ implies $t(s) \to \infty$), and only a finite number of $\alpha_1, \cdots, \alpha_n$, so for every $i = 1, \cdots, n$ two different t_i', t_i'' with $\alpha_i^{t_i'} = \alpha_i^{t_i''}$ exist. So $\alpha_i^{t_i' - t_i''} = 1$, that is: All α_i are roots of unity. Now choose s, and *a fortiori* $t = t(s)$, as a common multiple of the

root of unity exponents of $\alpha_1, \cdots, \alpha_n$. Then $\alpha_1^t = \cdots = \alpha_n^t = 1$, and as $\alpha_1, \cdots, \alpha_n$ are a permutation of $\alpha_1^t, \cdots, \alpha_n^t$, therefore $\alpha_1 = \cdots = \alpha_n = 1$.

So all characteristic values of U are equal to 1, and consequently $U = 1$.

From this we conclude immediately:

LEMMA 2. *Let $\mathcal{G}$ be a group, and $\mathcal{G}_0$ as described in §1 above. Consider a fixed $a \in \mathcal{G}$. Assume that there exists for every s $(= i, 2, \cdots)$ a $t = t(s)$ $(= i, 2, \cdots)$ which is an integral multiple of s, such that for a suitable $b_t \in \mathcal{G}$ there is $a^t = b_t a b_t^{-1}$. Then $a \in \mathcal{G}_0$.*

PROOF: Apply Lemma 1 to $U = D(a)$ and $V_t = D(b_t)$, where $D(x)$ is a finite dimensional, linear, unitary representation of $\mathcal{G}$. Then $D(a) = 1$, hence $a \in \mathcal{G}_0$ by (II) in §1.

5. We will now discuss the following groups as to their types of almost periodicity:

(α) $\mathcal{G} = \mathcal{G}^{(\alpha)}$, the group of all linear transformations

$$a'(u, \quad v): \qquad x \to x' \equiv ux + v,$$

where $u \neq 0$, u, v rational.

(β) $\mathcal{G} = \mathcal{G}^{(\beta)}$, the group of all rational transformations

$$a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix}: \qquad x \to x'' \equiv \frac{ux + v}{wx + z}$$

where $\operatorname{Det}\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} = uz - vw = 1,$

u, v, w, z rational integers.

(γ) $\mathcal{G} = \mathcal{G}^{(\gamma)}$, same as above, but u, v, w, z must be rational.

Discussion of (α): Since

$$a'(s, 0)a'(1, v)a'(s, 0)^{-1} = a'(1, sv) = (a'(1, v))^s$$

$(s = 1, 2, \cdots)$, so Lemma 2 applies to $a'(1, v)$ (with $t = t(s) = s$). Hence $a'(1, v) \in \mathcal{G}_0$.

On the other hand $D_\lambda(a'(u, v)) = (\exp(2\pi i\lambda \ln |u|))$ is a finite (one!) dimensional, linear, unitary representation of $\mathcal{G}^{(\alpha)}$ for every real λ. If $u \neq 1$, then $D_\lambda(a'(u, v)) \neq 1$ for some suitable real λ, hence $a'(u, v) \notin \mathcal{G}_0$ in this case.

Thus $\mathcal{G}_0$ is the set of all $a'(1, v)$, and consequently $\mathcal{G}$ is neither maximally nor minimally almost periodic.

Discussion of (β): This is the *modular group*; we observe (without giving the—well known—proofs, since these facts are not essential in our deductions): $\mathcal{G}^{(\beta)}$ is generated by the two elements

$$a_1 = a''\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}: \qquad x \to x'' \equiv -\frac{1}{x},$$

$$a_2 = a''\begin{pmatrix} 0, & -1 \\ 1, & 1 \end{pmatrix}: \qquad x \to x'' \equiv -\frac{1}{x+1}.$$

We have $a_1^2 = a_2^3 = 1$, and these are the only relations which hold for a_1, a_2. So $\mathcal{G}^{(\beta)}$ is the free group of 2 generators of order 2 and 3.

The almost periodic character of $\mathfrak{G}^{(\beta)}$ is determined as follows:

For any $p = 2, 3, \cdots$ the $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix}$ with $\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \cdots \bmod p$ form an invariant subgroup $\mathfrak{G}'_p$ of $\mathfrak{G}$. Two $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix}$ and $a''\begin{pmatrix} \bar{u}, & \bar{v} \\ \bar{w}, & \bar{z} \end{pmatrix}$ belong to the same coset of $\mathfrak{G}'_p$ in $\mathfrak{G}$ if and only if $\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \equiv \begin{pmatrix} \bar{u}, & \bar{v} \\ \bar{w}, & \bar{z} \end{pmatrix} \cdots \bmod p$. Hence the number of these cosets is finite, i.e.: the index of $\mathfrak{G}'_p$ in $\mathfrak{G}$ is finite.

Therefore if $a = a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \epsilon\, \mathfrak{G}_1$ (cf. §2 above), then $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \epsilon\, \mathfrak{G}'_p$ for all $p = 2, 3, \cdots$, i.e.: $\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \cdots \bmod p$ for all $p = 2, 3, \cdots$. Consequently $\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$, $a = 1$. Hence $\mathfrak{G}_1 = (1)$, and so *a fortiori* $\mathfrak{G}_0 = (1)$ (cf. §2 above). Thus $\mathfrak{G}^{(\beta)}$ is maximally almost periodic.

Discussion of (γ): Since

$$a''\begin{pmatrix} s, & 0 \\ 0, & \frac{1}{s} \end{pmatrix} a''\begin{pmatrix} 1, & v \\ 0, & 1 \end{pmatrix} a''\begin{pmatrix} s, & 0 \\ 0, & \frac{1}{s} \end{pmatrix}^{-1} = a''\begin{pmatrix} 1, & s^2 v \\ 0, & 1 \end{pmatrix} = \left(a''\begin{pmatrix} 1, & v \\ 0, & 1 \end{pmatrix}\right)^{(s^2)},$$

$(s = i, 2, \cdots)$, so Lemma 2 applies to $a''\begin{pmatrix} 1, & v \\ 0, & 1 \end{pmatrix}$ (with $t = t(s) = s^2$). Hence $a''\begin{pmatrix} 1, & v \\ 0, & 1 \end{pmatrix} \epsilon\, \mathfrak{G}_0$.

Since $\mathfrak{G}_0$ is an invariant subgroup of $\mathfrak{G}$, we can now infer successively that the following further elements belong to $\mathfrak{G}_0$:

$$a''\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} a''\begin{pmatrix} 1, & -v \\ 0, & 1 \end{pmatrix} a''\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}^{-1} = a''\begin{pmatrix} 1, & 0 \\ v, & 1 \end{pmatrix}$$

$$a''\begin{pmatrix} 1, & v_1 \\ 0, & 1 \end{pmatrix} a''\begin{pmatrix} 1, & 0 \\ v_2, & 1 \end{pmatrix} a''\begin{pmatrix} 1, & v_3 \\ 0, & 1 \end{pmatrix} = a''\begin{pmatrix} 1 + v_1 v_2, & v_1 + v_3 + v_1 v_2 v_3 \\ v_2 & 1 + v_2 v_3 \end{pmatrix}.$$

The general $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix}$ with $uz - vw = 1$; u, v, w, z rational, has the above form if $w \neq 0$: For this purpose we must determine v_1, v_2, v_3 with

$$v_2 = w, \qquad 1 + v_1 v_2 = u, \qquad 1 + v_2 v_3 = z, \qquad v_1 + v_3 + v_1 v_2 v_3 = v.$$

The three first equations can be satified directly by choosing v_2, v_1, v_3 respectively. The last one then holds automatically, since $uz - vw = 1$. So $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \epsilon\, \mathfrak{G}_0$ when $w \neq 0$. In particular $a''\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} \epsilon\, \mathfrak{G}_0$. Since

$$a''\begin{pmatrix} -v, & w \\ -z, & -w \end{pmatrix} a''\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} = a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix},$$

the same can be asserted when $z \neq 0$. And as $uz - vw = 1$ excludes $w = z = 0$, we have thus $a''\begin{pmatrix} u, & v \\ w, & z \end{pmatrix} \in \mathfrak{G}_0$ unrestrictedly.

Hence $\mathfrak{G}_0 = \mathfrak{G}$, and consequently $\mathfrak{G}^{(\gamma)}$ is minimally almost-periodic.

It seems to be worth emphasizing that the fundamental difference between the groups $\mathfrak{G}^{(\beta)}$ and $\mathfrak{G}^{(\gamma)}$ is brought about by requiring u, v, w, z to be rational only, instead of requiring them to be rational integers.

As $\mathfrak{G}^{(\beta)}$ is a subgroup of $\mathfrak{G}^{(\gamma)}$, the interesting situation arises that there are representations of $\mathfrak{G}^{(\beta)}$ which are not contained in any representation of $\mathfrak{G}^{(\gamma)}$. It is well known that this could not occur for finite or compact groups.

We mention finally that the minimal almost-periodicity of the common Lorentz group can be readily inferred from earlier work.[2]

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[2] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Annals of Math., 40 (1939), pp. 149–204. Cf. in particular pp. 164–168.

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NOTE ON THE RELATIVE STRUCTURE OF p-ADIC FIELDS

BY SAUNDERS MAC LANE

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This paper contains the solution of a problem stated and partially solved in a previous paper[1] on fields with discrete valuations. It also contains a correction of certain results given previously for non-discrete valuations.

1. The Problem of Relative Structure

A *p-adic field* K is a field of characteristic zero, complete under a valuation function $V(a)$ with integral values, such that the rational prime p has the value $V(p) = 1$. The structure theorem states that such a field K is uniquely determined (up to an analytic isomorphism) by the field $\mathfrak{K}$ of its residue classes (mod p). The original proof of Hasse and Schmidt for this structure theorem involved certain difficulties[2] with Steinitz towers of fields, but recently a different analysis of these towers has been found which re-establishes[3] a modified form of the Hasse-Schmidt proof.

Let K and K' be two p-adic extensions of a given p-adic field k, both with the same residue class field $\mathfrak{K}$. (Then $\mathfrak{K}$ is an extension of the residue class field $\mathfrak{k}$ of the original k). K and K' are *analytically equivalent* over $\mathfrak{K}$ and k if there is an analytic isomorphism which maps K on K' and leaves each element of k and each residue class of $\mathfrak{K}$ fixed. The relative structure problem is that of finding conditions on the residue class extension $\mathfrak{K}/\mathfrak{k}$ under which any two extensions K and K' are analytically equivalent. It suffices (*loc. cit.*, Theorem 8) to assume that $\mathfrak{K}$ "preserves p-independence" in $\mathfrak{k}$, in the sense defined later. This condition is also necessary.

THEOREM. *Let $\mathfrak{K}$ be any extension of the residue class field $\mathfrak{k}$ of a given p-adic field k. A necessary and sufficient condition that any two p-adic extensions of k with residue class field $\mathfrak{K}$ be analytically equivalent over $\mathfrak{K}$ and k is the requirement that $\mathfrak{K}/\mathfrak{k}$ preserve p-independence.*

Our proof that this requirement is necessary is an extension of a previously given example[1] of two non-equivalent extensions for the special case when $\mathfrak{K}$ is an inseparable algebraic extension of $\mathfrak{k}$. The employment of this type of example requires that we first (Lemma 1) break up an arbitrary extension $\mathfrak{K}/\mathfrak{k}$ which does not preserve p-independence into parts, one of which is an inseparable extension.

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[1] S. MacLane, *Subfields and Automorphism groups of p-adic fields*, these Annals, vol. 40(1939), pp. 423–442. Subsequently referred to as "subauto."

[2] S. MacLane, *Steinitz field towers for modular fields*, Transactions of the American Mathematical Society, vol. 46(1939), pp. 23–45.

[3] See a forthcoming paper *Über inseparable Körper*, by F. K. Schmidt and S. MacLane, in Mathematische Zeitschrift.

2. Extensions Preserving p-Independence

All residue class fields of p-adic fields necessarily have characteristic p. In a filed $\mathfrak{K}$ of this characteristic a subset X is said to be *p-independent* if no element of X lies in the field obtained by adjoining all the other elements of X to $\mathfrak{K}^p$, the field of all p^{th} powers of elements in $\mathfrak{K}$. An extension $\mathfrak{K}/\mathfrak{k}$ is said to *preserve p-independence*[4] if every subset of $\mathfrak{k}$ p-independent in $\mathfrak{k}$ remains p-independent in $\mathfrak{K}$.

LEMMA 1. *An extension $\mathfrak{K}$ of a field $\mathfrak{k}$ of characteristic p fails to preserve p-independence if and only if $\mathfrak{K}$ contains a subfield $\mathfrak{L} \supset \mathfrak{k}$ and an element η not in $\mathfrak{L}$ such that η^p lies in a field $\mathfrak{L}^p(\gamma_1, \cdots, \gamma_n)$ generated over $\mathfrak{L}^p$ by elements γ_i from $\mathfrak{k}$.*

Given such an $\mathfrak{L}$ and η, one readily verifies that certain elements γ p-independent in $\mathfrak{k}$ must become p-dependent in $\mathfrak{K}$. Conversely, if $\mathfrak{K}/\mathfrak{k}$ does not preserve p-independence we can find a finite set $\Gamma = [\gamma_1, \cdots, \gamma_n]$ of elements of $\mathfrak{k}$ p-independent in $\mathfrak{k}$ but not in $\mathfrak{K}$. Γ is then also p-dependent in some subfield of $\mathfrak{K}$ generated over $\mathfrak{k}$ by a finite number of elements. Let us add these elements to $\mathfrak{k}$ one at a time, taking care to make each such adjunction either a transcendental extension, a separable algebraic extension, or the addition of a p^{th} root. There is then some one of these elements η, the adjunction of which to the previously obtained field $\mathfrak{L}$ first brings about the p-dependence of Γ. This element η can be neither transcendental nor separable algebraic over $\mathfrak{L}$, for an extension of either of these types is known[1] to preserve p-independence. Therefore η is a p^{th} root of some element of $\mathfrak{L}$.

By definition the p-dependence of Γ in $\mathfrak{L}(\eta)$ means that some one element of Γ, say γ_1, can be written as a polynomial

$$(1) \qquad \gamma_1 = f(\gamma_2, \cdots, \gamma_n, \eta^p)$$

with coefficients in $\mathfrak{L}^p$, and with degree at most $p - 1$ in η^p or in any γ_i. Since Γ was not p-dependent in the field $\mathfrak{L}$, this polynomial actually involves η^p. Therefore η^p satisfies the separable equation (1) over $\mathfrak{L}^p(\gamma_1, \cdots, \gamma_n)$. But η^p also satifies over $\mathfrak{L}^p$ a purely inseparable equation, for $(\eta^p)^p$ is in $\mathfrak{L}^p$. Therefore η^p lies in $\mathfrak{L}^p(\gamma_1, \cdots, \gamma_n)$, as asserted.

Suppose again that $\mathfrak{K}/\mathfrak{k}$ does not preserve p-independenee, and return to the proof of the Theorem. The conclusion of the Lemma states that η can be written as

$$(2) \qquad \eta^p = \alpha_1^p\beta_1 + \alpha_2^p\beta_2 + \cdots + \alpha_m^p\beta_m, \qquad \alpha_i \text{ in } \mathfrak{L},$$

where each β_i is a power product of elements γ_j, and hence is in $\mathfrak{k}$. Extend the given p-adic field k to obtain[5] two p-adic fields L and L', both with the residue class field $\mathfrak{L}$. For each residue class β_i of (2) choose in k some representative b_i, and for each α_i choose representatives a_i and a_i' in L and L', respectively. Since the residue class equation (2) is an irreducible equation of degree p in η, we know that corresponding equations over L and L' must be irreducible. Let y and y' be defined by two such equations,

[4] A discussion of such extensions appears in S. MacLane, *Modular fields. I. Separating transcendence bases.* Duke Mathematical Journal, vol. 5(1939), pp. 372–393.

[5] Such an extension of a p-adic field is possible, by Theorem 1 of "subauto."

$$(3) \qquad y^p = \sum_i a_i^p b_i \qquad y'^p = (1 + p)(\sum_i a_i'^p b_i).$$

These elements generate p-adic fields $L(y)$ and $L'(y')$. Since y and y' both have the residue class η defined by (2), these fields both have the residue class field $\mathfrak{L}(\eta)$. Again extend[5] these fields $L(y)$ and $L'(y')$ to the p-adic fields K and K', respectively, with the extended residue class field $\mathfrak{K}$.

We assert that these two p-adic extensions K and K' are not analytically equivalent over $\mathfrak{K}$ and k. For suppose instead that K and K' were equivalent under some isomorphism $a \leftrightarrow a^T$. Since T is to leave k and residue classes of $\mathfrak{K}$ fixed, we know that

$$b_i^T = b_i, \qquad a_i^T \equiv a_i'(\text{mod } p), \qquad y^T \equiv y'(\text{mod } p).$$

According to Lemma 1 of "subauto," this implies

$$(a_i^p)^T \equiv a_i'^p(\text{mod } p^2), \qquad (y^p)^T \equiv y'^p(\text{mod } p^2).$$

Substitution of these in the first equation of (3) and subtraction of the second gives

$$p(\sum_i a_i'^p b_i) \equiv 0 \text{ (mod } p^2), \qquad \sum_i a_i'^p b_i \equiv 0 \text{ (mod } p),$$

a contradiction, for the residue class η of $\sum a_i'^p b_i$ is not 0. The existence of the two non-equivalent fields completes the solution of the relative structure problem, as stated in the Theorem.

3. A Correction

H. L. Kaplansky has called to my attention the fact that Theorems 3 and 4 of "subauto" cannot be correct without restriction in the case of non-discrete valuations. The difficulty arises because a non-discrete complete field L can have a proper complete extension L which has the same value group and residue class field as does L. Then K is an *immediate* extension of L in the sense of W. Krull,[6] who observed this phenomenon in connection with power series fields.

In the proof of Theorem 4, this difficulty appears in the argument that $K^{T*} = K'$ (p. 428 of "subauto"). The same trouble arises in Theorem 3. Both Theorems remain valid for discrete valuations, so the subsequent Theorems of the paper, which treat only this case, are not affected. The argument given for Theorem 3 in the non-discrete case does in fact prove a modified Theorem 3* which asserts, not that T^* is an analytic isomorphism of K on a subfield of K', but that T^* is an analytic isomorphism of some *subfield* K_0 of K on a subfield of K', where K_0 and K have the common residue class field $\mathfrak{K}$.

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[6] W. Krull, *Allgemeine Bewertungstheorie*, Journal für die Mathematik, vol. 167(1932), pp. 160–196.

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ON THE RATIONAL AUTOMORPHS OF $x_1^2 + x_2^2 + x_3^2$

BY GORDON PALL

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1. *Notations.* References followed by Q refer to an associated article.[1] We use the notations of §1Q. In addition to the sets $\mathfrak{Q}$, $\mathfrak{C}$, $\mathfrak{M}$ there defined, we employ German capitals for the following sets:

$\mathfrak{L}$: the 48 automorphs obtained from any one by shuffling rows;
$\mathfrak{A}$: all automorphs obtained from a given one by shuffling rows and shuffling columns;
$\mathfrak{C}$: the pure quaternion residues (mod m) obtained from a set $\mathfrak{M}$ by shuffling v_1, v_2, v_3;
$\mathfrak{K}$: all pure quaternions obtained from a pure one x by shuffling x_1, x_2, x_3.

Here *shuffling* denotes "permuting and changing signs of." In §3, letters which elsewhere represent integers denote real numbers.

We shall establish a one-to-one-to-one interconnection between the rational automorphs of $x_1^2 + x_2^2 + x_3^2$ and certain sets of solutions of (1_1Q) and (1_2Q). Numerous arithmetical properties of the automorphs and some additional properties of quaternions are obtained.

2. The rational automorphs of $x_1^2 + x_2^2 + x_3^2$ are the matrices

$$A = (a_{\alpha\beta}/m) \tag{1}$$

$$(\alpha, \beta = 1, 2, 3; \gcd (a_{11}, a_{12}, \cdots, a_{33}, m) = 1; m > 0)$$

such that, if A^* denotes the transpose matrix and I the identity,

$$A^*A = I = AA^*, \qquad A^* = A^{-1}. \tag{2}$$

Here $|A| = \theta = \pm 1$, and the relations (2) expand into the following:

$$\sum_\beta a_{\alpha\beta}^2 = \sum_\beta a_{\beta\alpha}^2 = m^2, \qquad \sum_\beta a_{\alpha\beta}a_{\gamma\beta} = 0 = \sum_\beta a_{\beta\alpha}a_{\beta\gamma} \quad \text{if } \alpha \neq \gamma, \tag{3}$$

$$\text{the cofactor of each element } a_{\alpha\beta} \text{ in } (a_{\alpha\beta}) \text{ is } \theta m a_{\alpha\beta}. \tag{4}$$

If m could be even (3_1) would imply that every $a_{\alpha\beta}$ is even. Similarly no prime factor $4f + 3$ of m can divide any $a_{\alpha\beta}$.

THEOREM 1. *The denominator m of any automorph* (1) *is odd. Each row and column of* $(a_{\alpha\beta})$ *satisfies*

$$x_1^2 + x_2^2 + x_3^2 = m^2, \tag{5}$$

[1] G. Pall, *On the Arithmetic of Quaternions*, Trans. Amer. Math. Soc., vol. 47 (1940), pp. 487–500. This article was originally intended to precede the present article in these Annals, but was transferred to the Transactions.

(6) *the g.c.d. of x_1, x_2, x_3 being* 1 *or a product of primes* $4f + 1$.

Trivially, two x_α in (5) *are even and one is odd. If* $m > 0$,

(7) *the even x_α in* (5) *are* $\equiv 0$ *if* $m \equiv 1$, $\equiv 2$ *if* $m \equiv 3 \pmod 4$.

In §8Q we proved a generalization of the fact that

(8) $x_1 = t_0^2 + t_1^2 - t_2^2 - t_3^2, \quad x_2 = 2(-t_0t_3 + t_1t_2), \quad x_3 = 2(t_0t_2 + t_1t_3)$

is the general solution of (5)–(6) with x_1 odd, t being a proper quaternion of norm m. For the purpose of proving (7), since any common factor of the x_α is $\equiv 1$ (4), it will suffice to show that every proper solution of (5) with x_1 odd is given by (8) for a proper t. Since $x = i_1x_1 + i_2x_2 + i_3x_3$ is proper and $Nx = m^2$, $x = vt$ with $Nt = m$ by Theorem 1Q, $Nv = m$, $v = \bar{l}a$ with $Na = 1$ since $\bar{x} = -x$ has t for a left divisor, whence $x = \bar{t}at$; x_1 being odd and $\bar{t}at \equiv (Nt)a \pmod 2$, $a = \pm i_1$; the case $a = -i_1$ reduces to $a = i_1$, since $\bar{t}(-i_1)t = \bar{u}i_1u$ if $t = i_2u$. Expanding $x = \bar{t}i_1t$ gives us (8). Finally, (7) follows on considering (8) with one or three of the t_i odd.

An automorph will be called *odd* if

(9) $$|A| = 1, \text{ and } a_{11}, a_{22}, a_{33} \text{ are odd.}$$

A class $\mathfrak{L}$ contains four odd automorphs obtainable from each other by changing signs of two rows.

3. The matrix function $\mathfrak{A}(t)$ of a *real* quaternion t, defined by

(10) $$\mathfrak{A}(t) = \frac{1}{Nt}\begin{bmatrix} t_0^2 + t_1^2 - t_2^2 - t_3^2 & 2(-t_0t_3 + t_1t_2) & 2(t_0t_2 + t_1t_3) \\ 2(t_0t_3 + t_1t_2) & t_0^2 - t_1^2 + t_2^2 - t_3^2 & 2(-t_0t_1 + t_2t_3) \\ 2(-t_0t_2 + t_1t_3) & 2(t_0t_1 + t_2t_3) & t_0^2 - t_1^2 - t_2^2 + t_3^2 \end{bmatrix}$$

is considered in this section. By the homogeneity,

(11) $$\mathfrak{A}(\lambda t) = \mathfrak{A}(t) \text{ for any real number } \lambda \neq 0.$$

If a matrix $B = (b_{\alpha\beta})$ is of the form $\mathfrak{A}(t)$ for some real quaternion t, then t is unique up to a factor λ. For by choice of λ we can suppose Nt to have any value $m > 0$. Equating (10) to $(b_{\alpha\beta}) = (a_{\alpha\beta}/m)$ we get the ten equations

(12) $4t_0^2 = m + a_{11} + a_{22} + a_{33}, \qquad 4t_1^2 = m + a_{11} - a_{22} - a_{33}, \cdots,$

(13) $4t_0t_1 = a_{32} - a_{23}, \qquad 4t_2t_3 = a_{23} + a_{32}, \cdots,$

which determine t_ft_g $(f, g = 0, 1, 2, 3)$ and hence an unique $\pm t$.

If further m and the $b_{\alpha\beta}$ are rational, every t_f^2 and t_ft_g is rational; $t_f = u_fn^{\frac{1}{2}}$ $(f = 0, 1, 2, 3)$ with rational u_f and n, $B = \mathfrak{A}(u)$. Choice of a factor λ makes u proper. Hence we have

LEMMA 1. *If a matrix B with rational elements is of the form $\mathfrak{A}(u)$ for some real quaternion u, then there are two and only two proper integral quaternions, t and $-t$, such that* $\mathfrak{A}(t) = B$.

The matrix $\mathfrak{A}(t)$ has the multiplicative property

$$\mathcal{A}(t)\cdot\mathcal{A}(u) = \mathcal{A}(tu). \tag{14}$$

This can be verified as follows. Let x denote either the

(15) pure quaternion $i_1x_1 + i_2x_2 + i_3x_3$, or matrix (x_α) of one column;

similarly for y. The columns of $Nt\mathcal{A}(t)$ are $ti_\alpha\bar{t}$ $(\alpha = 1, 2, 3)$. Hence $Nt\mathcal{A}(t)x$ corresponds to $\sum x_\alpha ti_\alpha\bar{t} = t(\sum x_\alpha i_\alpha)\bar{t} = tx\bar{t}$, that is, if $A = \mathcal{A}(t)$ the matrix equation

$$Ax = y \tag{16}$$

corresponds to the quaternion equation

$$tx\bar{t} = my, \text{ where } m = Nt. \tag{17}$$

Hence (14) follows when we observe that for arbitrary x and y,

$$t(ux\bar{u})\bar{t}/(NuNt) = y \textit{ is equivalent to } (tu)x(\overline{tu})/N(tu) = y,$$

$$\mathcal{A}(t)\mathcal{A}(u)x = y \textit{ is equivalent to } \mathcal{A}(tu)x = y.$$

For any non-zero real quaternion t, $\mathcal{A}(t)$ is a real automorph of $x_1^2 + x_2^2 + x_3^2$; for by taking norms in (17), $Nx = Ny$. Also $|\mathcal{A}(t)|$ is $+1$, and not -1, for every t, since by continuous transformation of t we can reach $t = \pm 1$ when $\mathcal{A}(t)$ is the identity matrix. It is worth noting the following identity in the x_α and t_i, the expressions in the matrix of (10) being substituted for the $a_{\alpha\beta}$:

$$(x_1^2 + x_2^2 + x_3^2)(t_0^2 + t_1^2 + t_2^2 + t_3^2)^2 = \sum_\alpha (a_{\alpha 1}x_1 + a_{\alpha 2}x_2 + a_{\alpha 3}x_3)^2.$$

We now prove conversely that every real automorph of $x_1^2 + x_2^2 + x_3^2$, with determinant $+1$, is of the form $\mathcal{A}(t)$ for real t; and it will follow from lemma 1 that every rational automorph is of that form for proper t.

It suffices to prove that if $m > 0$ and (3)–(4) hold with $\theta = 1$, the ten equations (12)–(13) are solvable in real t_i. By (3_1) and (4),

$$a_{32}^2 - a_{23}^2 = a_{13}^2 - a_{31}^2 = a_{21}^2 - a_{12}^2 \ (= \varepsilon, \text{ say}), \tag{18}$$

$$ma_{12} = a_{23}a_{31} - a_{21}a_{33}, \qquad ma_{21} = a_{13}a_{32} - a_{12}a_{33}, \text{ etc. cyclically.} \tag{19}$$

Thus $a_{12} = \pm a_{21}$ implies $a_{23}a_{31} = \pm a_{13}a_{32}$ with the same sign; and similarly on permuting subscripts cyclically. Hence, if $\varepsilon = 0$:

a) we can set $a_{23} = \eta_1 a_{32}$ etc., each $\eta_\alpha = \pm 1$, $\eta_1\eta_2\eta_3 = 1$;

b) if a_{23}, a_{31}, or a_{12} vanishes, at least two of them vanish.

Case I, $\varepsilon = 0$, at least two of a_{23}, a_{31}, a_{12} zero; say a_{31} and a_{12}. Then $a_{23} = \pm a_{32}$. For the $+$ sign, (3) and $|A| > 0$ imply $a_{33} = -a_{22}$, $a_{11} = -m$; take $t_0 = t_1 = 0$, $2t_2t_3 = a_{23}$, $t_2^2 + t_3^2 = m$; then $a_{22}^2 = m^2 - a_{23}^2 = (t_2^2 - t_3^2)^2$, and by permuting t_2, t_3, $a_{22} = t_2^2 - t_3^2$. The rest of (12)–(13) follows. If $a_{23} = -a_{32}$, $t_2 = t_3 = 0$ yields a similar result.

CASE II, $\varepsilon = 0$, no $a_{\alpha\beta} = 0 (\alpha \neq \beta)$. According to the cases 0) $\eta_1 = \eta_2 = \eta_3 = 1$, 1) $\eta_2 = \eta_3 = -1$, 2) $\eta_3 = \eta_1 = -1$, 3) $\eta_1 = \eta_2 = -1$, we take t_0, t_1, t_2, or t_3 to be zero. Three of equations (13) become trivial, the rest determine an unique $\pm t$, and imply respectively: 0) $2t_1^2 = a_{31}a_{12}/a_{23}, \cdots$; 1) $2t_0^2 = a_{21}a_{13}/a_{23}$, $2t_2^2 = a_{13}a_{23}/a_{21}$, $2t_3^2 = a_{21}a_{23}/a_{13}$; and similarly in cases 2) and 3). Equations (12) now follow. For example in case 0), by (4) and (3_2), $a_{12}(m + \sum a_{\alpha\alpha}) = a_{12}a_{11} + a_{12}a_{22} + a_{12}(m + a_{33}) = a_{12}a_{11} + a_{22}a_{21} + a_{31}a_{32} = 0$ $= 4t_0^2 a_{12}$, $a_{23}(m + a_{11} - a_{22} - a_{33}) = a_{21}a_{31} - a_{22}a_{32} - a_{23}a_{33} = 2a_{21}a_{31} = 4t_1^2 a_{23}$, etc. In case 1), $a_{23} = a_{32}$, $a_{31} = -a_{13}$, and $a_{12} = -a_{21}$, whence for example, $a_{23}(m + \sum a_{\alpha\alpha}) = a_{12}a_{31} + a_{23}a_{22} + a_{23}a_{33} = -a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = -2a_{21}a_{31}$ $= 4t_0^2 a_{23}$.

CASE III, $\varepsilon \neq 0$. Then all of (13) are implied by the conditions $16t_0t_1t_2t_3 = \varepsilon$, $4t_2t_3 = a_{23} + a_{32}$, $4t_3t_1 = a_{31} + a_{13}$, $4t_1t_2 = a_{12} + a_{21}$, which determine $\pm t$ uniquely. Also (12) follow. For example by (19),

$$(a_{12} + a_{21})(m + a_{11} + a_{22} + a_{33}) = a_{13}a_{32} + a_{31}a_{23} + a_{21}a_{11} + a_{12}a_{22} + a_{21}a_{22} + a_{12}a_{11}$$
$$= (a_{32} - a_{23})(a_{13} - a_{31}) = 4t_0^2(a_{21} + a_{12}) \text{ by (13).}$$

4. THEOREM 2. *A rational automorph $A = (a_{\alpha\beta}/m)$ of denominator m and determinant $+1$ is of the form $\mathfrak{A}(t)$ for an unique pair of proper quaternions $\pm t$; Nt is m, $2m$, or $4m$ according as A contains three, one, or no odd $a_{\alpha\alpha}$.*

The first part follows from §3. If in (10) the denominator reduces to m, $Nt = hm$ for some integer h dividing all nine elements of the matrix $Nt\mathfrak{A}(t)$. Since t is proper and obvious combinations of the diagonal elements with Nt produce $4t_f^2 (f = 0, 1, 2, 3)$, $h = 1, 2$, or 4. Conversely if t is proper and Nt is m, $2m$, or $4m$ (m odd), the denominator to which $\mathfrak{A}(t)$ reduces is indeed m; for any prime dividing m and the three diagonal terms divides each t_i. The possible parities of the t_i in each case show that three $a_{\alpha\alpha}$ are odd if $Nt = m$, one is odd if $Nt = 2m$, and all even if $Nt = 4m$, every t_i odd.

THEOREM 3. *Let u be proper, m odd. If $Nu = 2m$, $\mathfrak{A}(u)$ can be derived from an odd A by interchanging two rows and changing the signs of one row. If $Nu = 4m$, $\mathfrak{A}(u)$ is obtainable from an odd A by permuting the rows cyclically.*

For if $2 \mid Nu$, the u_i are congruent (mod 2) in pairs. Hence $u = (1 + i_\alpha)t$ with t integral, $\alpha = 1, 2$, or 3; $Nt = \frac{1}{2}Nu$, $\mathfrak{A}(u) = \mathfrak{A}(1 + i_\alpha)\mathfrak{A}(t)$; and

$$\mathfrak{A}(1 + i_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The case $Nu = 4m$ is solved by two applications of this process, and

LEMMA 2. *The three automorphs obtained from $\mathfrak{A}(t)$ by changing the signs of two of its rows are $\mathfrak{A}(i_1t)$, $\mathfrak{A}(i_2t)$, $\mathfrak{A}(i_3t)$.*

The four odd automorphs of a class $\mathfrak{L}$ (end §2) are, in view of Theorem 2 and lemma 2, associated with an unique set $\mathfrak{Q}$ (§1Q).

We denote the set of quaternions conjugate to those of $\mathfrak{E}$ by $\mathfrak{E}^*$; the set of automorphs transpose to those of $\mathfrak{A}$ by $\mathfrak{A}^*$. Thus $\mathfrak{E}^* = \mathfrak{E}$ if and only if

$$\text{(20)} \qquad \text{an equality occurs among } t_0^2,\ t_1^2,\ t_2^2,\ t_3^2,\ 0.$$

THEOREM 4. *As t ranges over a set $\mathfrak{E}$ (or $\mathfrak{Q}$) of odd norm m, $\mathfrak{A}(t)$ ranges twice, as $\mathfrak{A}(t) = \mathfrak{A}(-t)$, over the odd automorphs of a set $\mathfrak{A}$(or $\mathfrak{L}$) of denominator m. If $\mathfrak{A}$ corresponds in this way to $\mathfrak{E}$, $\mathfrak{A}^*$ corresponds to $\mathfrak{E}^*$.*

The proof for $\mathfrak{Q}$ and $\mathfrak{L}$ was given above. We can restrict t to one value in every subset $\mathfrak{Q}$ of $\mathfrak{E}$, say that given in lemma 13Q. By forming the automorphs $\mathfrak{A}(\eta)$, for η in (15Q), we find that $\mathfrak{A}(\eta t\bar{\eta}) = \mathfrak{A}(\eta)\mathfrak{A}(t)\mathfrak{A}(\bar{\eta})$ is obtained by the following respective operations:

$$\text{(21)} \qquad \begin{aligned} &\text{identity}, \quad \sigma_\alpha, \quad \sigma_{\alpha+1}\pi_{\alpha+1,\alpha+2}, \quad \sigma_{\alpha+2}\pi_{\alpha+1,\alpha+2}, \quad \pi_{\alpha+1,\alpha+2}, \quad \sigma_\alpha\pi_{\alpha+1,\alpha+2}, \\ &\pi_{123}, \quad \pi_{123}\sigma_\alpha, \quad \pi_{321}, \quad \pi_{321}\sigma_\alpha; \qquad (\alpha = 1, 2, 3). \end{aligned}$$

Here the subscripts are to be reduced (mod 3) to 1, 2, 3; σ_α denotes the operation of changing the sign of the α-th row, then of the α-th column; $\pi_{\alpha\beta}$ indicates the interchange of the α-th and β-th rows, then of the α-th and β-th columns; π_{123} represents a cyclic permutation of rows, and then columns.

If an *odd* A' is derived by shuffling (§1) rows, and columns, from an odd A, any rearrangement of rows must be accompanied by the same rearrangement of columns, and since $|A| > 0$, the number of sign-changes must be even. All such possibilities, except for sign-changes of two rows, which are provided for in the $\mathfrak{L}$-classes, are expressed in (21). Theorem 4 follows, the last part being obvious from (10): $\mathfrak{A}(\bar{t}) = \mathfrak{A}^*(t)$.

It may be observed from the last part of §3 that some t_i vanishes if and only if $\varepsilon = 0$, and that then A becomes symmetric on changing signs of certain rows. From (10) we see that if $t_1 = t_2$, then $a_{11} = a_{22}$, $a_{31} = a_{23}$, $a_{13} = a_{32}$, and A becomes symmetric on interchanging the last two rows; and similarly if any equality occurs among $t_0^2, t_1^2, t_2^2, t_3^2, 0$. Conversely, by (10), if $a_{11} = a_{22}$, then $t_1^2 = t_2^2$; if $a_{12} = a_{31}$, then $t_0 = -t_1$ or $t_2 = t_3$; the possibility $a_{11} = \pm a_{23}$ implies $(t_0 \pm t_1)^2 = (t_2 \pm t_3)^2$ and is excluded by residues (mod 2) if Nt is odd. Thus we have two theorems:

THEOREM 5. *If some two elements not in the same row or column of A are numerically equal, then the class $\mathfrak{L}$ of A contains a symmetric automorph.*

THEOREM 6. *A class $\mathfrak{A}$ contains a symmetric automorph if and only if two of $t_0^2, t_1^2, t_2^2, t_3^2, 0$ are equal in the corresponding proper $\mathfrak{E}$.*

5. In view of the equivalence of (16) and (17), the identities

$$t(t - t_0)\bar{t} = (t - t_0)Nt, \qquad t(i_\alpha t + t_\alpha)\bar{t} = (ti_\alpha + t_\alpha)Nt,$$

give us the lemma and corollary:

LEMMA 3. *On multiplying by $\mathfrak{A}(t)$ on the left, the column vector (t_1, t_2, t_3) becomes (t_1, t_2, t_3), $(t_0, -t_3, t_2)$ becomes $(t_0, t_3, -t_2)$, $(t_3, t_0, -t_1)$ becomes $(-t_3, t_0, t_1)$, $(-t_2, t_1, t_0)$ becomes $(t_2, -t_1, t_0)$.*

COROLLARY 1. *If $\mathfrak{A}(t) = (a_{\alpha\beta}/Nt)$ in (10), then*

$$\sum_\beta a_{\alpha\beta} y_\beta \equiv 0 (\mathrm{mod}\ Nt) \ (\alpha = 1, 2, 3) \text{ for each of } (y_1, y_2, y_3)$$
$$= (t_1, t_2, t_3), (t_0, -t_3, t_2), (t_3, t_0, -t_1), (-t_2, t_1, t_0). \tag{22}$$

In fact (22) gives the identities in the proof of Theorem 5′Q. Hence

COROLLARY 2. *If $p^r \mid Nt$ and $p \nmid t_0^2 + t_\alpha^2$, the four congruences*

$$\begin{aligned} u_0t_0 - u_1t_1 - u_2t_2 - u_3t_3 &\equiv 0, & u_0t_1 + u_1t_0 + u_2t_3 - u_3t_2 &\equiv 0, \\ u_0t_2 - u_1t_3 + u_2t_0 + u_3t_1 &\equiv 0, & u_0t_3 + u_1t_2 - u_2t_1 + u_3t_0 &\equiv 0, \end{aligned} \tag{23}$$

obtained on expanding $ut \equiv 0$ (mod p^r), can be expressed as linear combinations of (23_1) *and* $(23_{\alpha+1})$.

6. THEOREM 7. *For any automorph* (1) *we can choose pure quaternions u and v such that*

$$a_{\alpha\beta} \equiv u_\alpha v_\beta \ (\mathrm{mod}\ m), \qquad \alpha, \beta = 1, 2, 3. \tag{24}$$

Here u and v must satisfy

$$Nu \equiv Nv \equiv 0, \ u \text{ and } v \text{ proper } (\mathrm{mod}\ m). \tag{25}$$

Also, u and v are uniquely determined (mod m) *except that we can replace* (u, v) *by* (eu, fv), *where e, f are any integers such that* $ef \equiv 1$ (mod m).

By the Chinese Remainder Theorem it suffices to determine u and v (mod p^r), for each p^r dividing m. Some $a_{\alpha\beta}$ is prime to p,[2] say a_{11}. Then let $u_1 = 1$, $v_\beta \equiv a_{1\beta}$, and determine u_2 and u_3 from $a_{21} \equiv u_2a_{11}$, $a_{31} \equiv u_3a_{11}$; (24) holds for every α and β, since by (4) every minor determinant of order 2 in $(a_{\alpha\beta})$ is divisible by p^r.

Since $m, a_{11}, a_{12}, \cdots, a_{33}$ are coprime, u and v must be proper; this with (3_1) implies that $m \mid Nu$ and Nv.

If $u_\alpha v_\beta \equiv u'_\alpha v'_\beta$ (mod m), $(\alpha, \beta = 1, 2, 3)$, we can find integers r_α, s_β such that $\sum r_\alpha u_\alpha \equiv 1 \equiv \sum s_\beta v_\beta$ (mod m). Set $e = \sum s_\beta v'_\beta$, $f = \sum r_\alpha u'_\alpha$. Then $u_\alpha \equiv \sum u_\alpha v_\beta s_\beta \equiv \sum u'_\alpha v'_\beta s_\beta \equiv eu'_\alpha$, $v_\beta \equiv fv'_\beta$, and $ef \equiv \sum s_\beta v'_\beta \sum r_\alpha u'_\alpha \equiv \sum\sum r_\alpha s_\beta u'_\alpha v'_\beta \equiv \sum\sum r_\alpha s_\beta u_\alpha v_\beta \equiv 1$ (mod m).

If $B = (b_{\alpha\beta}/n)$ is an automorph of denominator n, and $m \mid n$, we write

$$B \sim v \ (\mathrm{mod}\ m) \tag{26}$$

to indicate that the three rows of $(b_{\alpha\beta})$ belong to the set $\mathfrak{M}$ (mod m) determined by v. By (24), $A \sim v$ and $A^* \sim u$ (mod m). Since u is proper (mod m) the set $\mathfrak{M}$ (mod m) containing all three rows of mA is evidently unique.

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[2] Examples with no $a_{\alpha\beta}$ prime to m may appear when m has three prime factors $4f + 1$. If $m = 5\cdot13\cdot17$, $v = 775i_1 + 51i_2 + 533i_3$ is effective for $\mathfrak{A}(t)$, $t = 28 + 11i_1 + 10i_2 + 10i_3$; likewise for $t = 24 + 22i_1 + 6i_2 + 3i_3$, every $a_{\alpha\beta}$ is divisible by 5, 13, or 17.

LEMMA 4. *If v is pure and proper* (mod m), *and $m \mid Nv$, we can secure*

$$m^2 \mid Nv \tag{27}$$

by adding multiples of m to v_1 and v_2.

Set $Nv = sm$, $w = hi_1 + ki_2$. Then $N(v + mw) = m(s + 2hv_1 + 2kv_2) + (h^2 + k^2)m^2$, and we can choose h and k to make $m \mid s + 2hv_1 + 2kv_2$, since v_1, v_2, m are coprime.

LEMMA 5. *If* (24) *holds for the automorph* (1), *and $m^2 \mid Nv$, the integers $h_{\alpha\beta}$ defined by $a_{\alpha\beta} = u_\alpha v_\beta + mh_{\alpha\beta}$ satisfy*

$$\sum h_{\alpha\beta}v_\beta \equiv 0 \text{ (mod } m), \ \alpha = 1, 2, 3. \tag{28}$$

For on substituting $a_{\alpha\beta} = u_\alpha v_\beta + mh_{\alpha\beta}$ in (3) and using (27) we get

$$m \mid u_\alpha \sum h_{\alpha\beta}v_\beta, \qquad m \mid u_\gamma \sum h_{\alpha\beta}v_\beta + u_\alpha \sum h_{\gamma\beta}v_\beta. \tag{29}$$

Multiply the latter by u_γ. Since m, u_α, u_γ^2 are coprime, (28) follows.

COROLLARY 3. *With the same hypotheses, $m^2 \mid \sum a_{\alpha\beta}v_\beta$.*

LEMMA 6. *If v is pure and proper, and Nv is odd,*

$$\mathfrak{A}(v) \sim v \text{ (mod } Nv). \tag{30}$$

For $\mathfrak{A}(v)$ is then of denominator Nv; (30) follows from (10) with t replaced by v, $v_0 = 0$: $a_{\alpha\beta} \equiv 2v_\alpha v_\beta$ (mod m).

We note here the similar fact that for proper t of odd norm,

$$\text{if } t_0 = t_1, \quad \mathfrak{A}(t) \sim 2t_1i_1 + (t_2 - t_3)i_2 + (t_2 + t_3)i_3 \text{ (mod } Nt); \tag{30'}$$

two like results being obtained by permuting subscripts 1, 2, 3 cyclically.

COROLLARY 4. *The preceding remarks furnish quickly a value of v for any symmetric automorph.*

LEMMA 7. *If x is proper and Nx odd, and $x = ut$, $Nt = m$, then the rows of $Nx\mathfrak{A}(x)$ are in the set $\mathfrak{M}$* (mod m) *containing the rows of $m\mathfrak{A}(t)$.*

For by (14), $Nx\mathfrak{A}(x) = Nu\mathfrak{A}(u) \cdot Nt\mathfrak{A}(t)$, whence the rows of $Nx\mathfrak{A}(x)$ are linear combinations with integer coefficients of the rows of $m\mathfrak{A}(t)$.

THEOREM 8. *Let v be pure and proper* (mod m), *$m \mid Nv$, t proper, $Nt = m$; then*

$$\mathfrak{A}(t) \sim v \text{ (mod } m) \textit{ if and only if } t \textit{ is a right divisor of } v. \tag{31}$$

By adding multiples of m to the v_α we make v actually proper and of odd norm; then (30) holds. I. Let $v = ut$. By lemma 7, if $\mathfrak{A}(t) \sim z(m)$, $\mathfrak{A}(v) \sim z(m)$. By (30), v and z are proportional (mod m), $\mathfrak{A}(t) \sim v(m)$. II. Conversely, let $\mathfrak{A}(t) \sim v(m)$. By lemmas 4 and 1Q we can make $m^2 \mid Nv$. Set $\mathfrak{A}(t) = (a_{\alpha\beta}/m)$, $a_{\alpha\beta} = u_\alpha v_\beta + mh_{\alpha\beta}$ as in lemma 5. Let $v = uy$, $Ny = m$. We must show that t and y are left-associates. By case I, $\mathfrak{A}(y) \sim v(m)$. Set $\mathfrak{A}(y) = (b_{\alpha\beta}/m)$, $b_{\alpha\beta} = w_\alpha v_\beta + mk_{\alpha\beta}$ as in lemma 5. Then $\mathfrak{A}(t\bar{y}) = \mathfrak{A}(t)\mathfrak{A}(y)^* = (c_{\alpha\gamma}/m^2)$, where

$$c_{\alpha\gamma} = \sum a_{\alpha\beta}b_{\gamma\beta} = u_\alpha w_\gamma \sum v_\beta^2 + mu_\alpha \sum k_{\gamma\beta}v_\beta + mw_\gamma \sum h_{\alpha\beta}v_\beta + m^2 \sum h_{\alpha\beta}k_{\gamma\beta},$$

is divisible by m^2. Hence $\mathfrak{A}(t\bar{y})$ has denominator 1, $t\bar{y} = m\eta$ with $N\eta = 1$, $t = \eta y$.

In (24) if A is odd, and corresponds to t, the vectors u and v are, respectively pure right and left multiples of t. By Theorem 3Q all left multiples (and similarly all right multiples) are proportional (mod m). By Theorem 9Q, u and v belong to the same set $\mathfrak{C}$ if and only if (20) holds.

If $A \sim v(m)$, and v' is obtained by shuffling v_1, v_2, v_3, and A' is obtained by the same shuffle of the columns of A, then $A' \sim v'(m)$. Theorems 4, 7, 8, and 4Q imply

THEOREM 9. *Every set* $\mathfrak{M}$ (mod m) *contains all three rows* ($\times m$) *of the automorphs in one and only one class* $\mathfrak{L}$ *of denominator* m, *and conversely; likewise for* $\mathfrak{C}$ *and* $\mathfrak{A}$.

We have thus, *for any odd positive* m, *a one-to-one-to-one association between sets* $\mathfrak{L}, \mathfrak{M}, \mathfrak{Q}$; *and* $\mathfrak{A}, \mathfrak{E}, \mathfrak{C}$.

7. THEOREM 10. *Let* x *be proper and of odd norm* m'', $m \mid m''$, $A'' = \mathfrak{A}(x)$. *The rows and columns of* $m''A''$ *are in the same set* $\mathfrak{C}$ (mod m) *if and only if*

(32) $$m \text{ divides one of } x_f,\ x_f \pm x_g (f \neq g),\ x_0 \pm x_1 \pm x_2 \pm x_3 .$$

For set $x = at$, $\bar{x} = bt'$, $Nt = m = Nt'$. By Theorem 8Q, (32) holds if and only if t and t' are in the same set $\mathfrak{E}$. The columns of $\mathfrak{A}(x)$ being the rows of $\mathfrak{A}(\bar{x})$, the theorem follows from lemma 7.

COROLLARY 5. *If* mA *is symmetrical* (mod m), *the class* $\mathfrak{L}$ *of* A *contains a symmetrical automorph.*

For the sets $\mathfrak{M}$ containing the rows and columns of mA coincide.

8. *Factorization of Automorphs.* We call A a *right divisor of* A'' if

(33) $$A'' = A'A, \text{ and } m'' = m'm \text{ holds for the denominators.}$$

Then every automorph in the set $\mathfrak{L}$ of A is a right divisor of every automorph in the set $\mathfrak{L}$ of A''.

LEMMA 8. *If* A *is a right divisor of* A'', *and* t *and* t'' *are in the corresponding sets* $\mathfrak{Q}$ *and* $\mathfrak{Q}''$, *then* t *is a right divisor of* t''.

For we can suppose A and A'' replaced by odd automorphs in their sets $\mathfrak{L}$, and have (33) with $A = \mathfrak{A}(t)$, $A'' = \mathfrak{A}(t'')$. A product of odd automorphs being obviously odd, we have $A' = A''A^* = \mathfrak{A}(t''\bar{t})$ of denominator m', whence $t''\bar{t} = \lambda t'$, $Nt' = m'$. By the norms $\lambda = \pm m$. Hence $t'' = \pm t't$.

LEMMA 9. *If* z *is proper* (mod m), *and* t *is a right divisor of* z *of norm* m, *then* $\mathfrak{A}(t)$ *is a right divisor of* $\mathfrak{A}(z)$.

For we can write $z = \lambda vy$, where λ is an integer prime to m, $Nv = 2^r$, y proper, Ny odd. Then $\mathfrak{A}(z)$ is in the class $\mathfrak{L}$ of $\mathfrak{A}(y)$ and is of denominator Ny. By Cor. 1'Q, the right divisors of y and z of norm m are the same. Hence $y = ut$, $\mathfrak{A}(t)$ is a right divisor of $\mathfrak{A}(y)$, hence of $\mathfrak{A}(z)$.

However, A need not be a right divisor of $A'A$, for the denominator of $A'A$ may be less than $m'm$. By shuffling rows of A, columns and rows of A', the

problem is reduced to the case where A' and A are odd, say $A' = \mathfrak{A}(u)$, $A = \mathfrak{A}(t)$. If ut is proper, A is a right divisor of $A'A$. This is trivially the case if m' and m are coprime (e.g. by lemma 9Q). We now have:

(a) The right divisors of denominator m of an automorph whose denominator is divisible by m, form an unique class $\mathfrak{L}$.

(b) An automorph of denominator $m_1 m_2 \cdots m_s$ (each m_ν odd) can be expressed in the form $A'A'' \cdots A^{(s)}$ where $A^{(\nu)}$ is of denominator $m_\nu (\nu = 1, \cdots, s)$, in essentially only one way; the general such expression being

$$(A'K'^*)(K'A''K''^*)(K''A'''K'''^*) \cdots (K^{(s-1)}A^{(s)}),$$

where the $K^{(i)}$ are integral automorphs. That is, the $K^{(i)}$ are matrices having one element ± 1 in each row and column, the rest 0; whence KAK' is in the class $\mathfrak{A}$ of A, and KA in the class $\mathfrak{L}$.

(c) If z is proper (mod m) and $m \mid Nz$, the right divisors of denominator m of $\mathfrak{A}(z)$ and $\mathfrak{A}(z + xm)$ are the same.

(d) If (33) holds, suppose $A'' \sim v$ (mod m''). By lemma 7, $A \sim v$ (mod m). Conversely, let A'' be of denominator $m'' = m'm$, $A'' \sim v(m'')$, and let $A \sim v(m)$. The right divisor of denominator m of A'' being also $\sim v(m)$, it is in the class $\mathfrak{L}$ of A. Hence A is a right divisor of A''.

It may be worth noting that if (33) holds, $m''A'' = (u_\alpha v_\beta) + m''(h_{\alpha\beta})$, and $(m'')^2 \mid Nu$ and Nv, as in lemma 5, and we set

$$mA = (s_\alpha v_\beta) + m(k_{\alpha\beta}), \qquad m'A' = (u_\alpha r_\beta) + m'(l_{\alpha\beta}),$$

then, as is easily seen by multiplying out $mA = mA'^*A''$,

$$s_\alpha \equiv (1/m') \sum u_\gamma l_{\gamma\alpha} \text{ (mod } m), \qquad r_\beta \equiv (1/m) \sum k_{\beta\alpha} v_\alpha \text{ (mod } m').$$

9. A natural application of automorphs is in transforming solutions of

$$x_1^2 + x_2^2 + x_3^2 = n \tag{34}$$

into other solutions. We employ the double interpretation (15) for x, y. If $A = (a_{\alpha\beta}/m)$ the equation $Ax = y$ expands into

$$\sum_\beta a_{\alpha\beta} x_\beta = m y_\alpha, \qquad \alpha = 1, 2, 3. \tag{35}$$

We say that A *is integrally effective on* x, if x and Ax are integral.

As A ranges over a set $\mathfrak{L}$, Ax ranges over a set $\mathfrak{K}$ (§1) precisely $2/k$ times, where k, called the *weight of* $\mathfrak{K}$, has the following values:

$$\begin{aligned} &k = 2, \textit{ if no two of } y_1^2, y_2^2, y_3^2, 0 \textit{ are equal;} \\ &k = 1 \textit{ if there is only one equality among } y_1^2, y_2^2, y_3^2, 0; \\ &k = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \textit{ resp. for the types } (g, g, 0), (g, g, g), (g, 0, 0). \end{aligned} \tag{36}$$

As A ranges over the four odd automorphs of a set $\mathfrak{L}$, Ax ranges over a set $[y]((23Q))$. We can then use the notation (17).

THEOREM 11. *Let $Nt = m$, t proper, m as always odd. Use the double notation* (15), *x integral but not necessarily proper. Then $\mathfrak{A}(t)$ is integrally effective on x, that is*

$$(txl)/m \text{ is integral,} \tag{37}$$

if and only if t is a right-divisor of $x_0 + x$ for some integer x_0.

Sufficiency. $x_0 + x = ut$, $tx\bar{t} = t(ut - x_0)\bar{t} = (tu - x_0)m$.

Necessity. Let $\mathfrak{A}(t) \sim v(\text{mod } m)$. By Theorem 8, $v = ut$. By Theorem 7, the condition that $\mathfrak{A}(t)$ be integrally effective on x is equivalent to

$$v_1x_1 + v_2x_2 + v_3x_3 \equiv 0 \ (\text{mod } m). \tag{38}$$

Corollary 7Q completes the proof. In place of cor. 7Q we may use

LEMMA 10. *Let m not have a factor in common with two of v_1, v_2, v_3. Every integral solution x of* (38) *is of the following form for certain integers w_1, w_2, w_3:*

$$x_1 \equiv w_2v_3 - w_3v_2, \qquad x_2 \equiv w_3v_1 - w_1v_3, \qquad x_3 \equiv w_1v_2 - w_2v_1 \ (\text{mod } m). \tag{39}$$

By the C. R. T. the proof reduces to modulus p^r. Let v_2 and v_3 be prime to p. Solve $w_2v_3 - w_3v_2 \equiv x_1(p^r)$ for w_2 and w_3; (38) becomes $v_3(x_3 + v_1w_2) \equiv v_2(v_1w_3 - x_2)$ (mod p). Hence $x_3 + v_1w_2 \equiv w_1v_2$ for a certain w_1, amd $w_1v_3 \equiv v_1w_3 - x_2$. Set $x_0 = \sum w_\alpha v_\alpha$. Then $x_0 + x \equiv wv = (wu)t$ (mod m).

THEOREM 12. *Let x be pure and proper* (mod m). *An automorph A of denominator m is integrally effective on x if and only if A is a right divisor of $\mathfrak{A}(x_0 + x)$ for some integer x_0.*

We can replace A by an odd automorph, $A = \mathfrak{A}(t)$, in its class $\mathfrak{L}$. If Ax is integral, $x_0 + x = ut$ for some x_0, by Theorem 11. By lemma 9, $\mathfrak{A}(t)$ is a right divisor of $\mathfrak{A}(x_0 + x)$. Conversely, if $\mathfrak{A}(t)$ is a right divisor of $\mathfrak{A}(x_0 + x)$, set $x_0 + x = \lambda vy$ as for lemma 9. Then $\mathfrak{A}(t)$ is a right divisor of $\mathfrak{A}(y)$, $y = ut$ by lemma 8, $x_0 + x = (\lambda vu)t$. By Theorem 11, $\mathfrak{A}(t)$ is integrally effective on x.

By lemma 1Q and corollary 6Q we have

LEMMA 11. *The $\mathfrak{L}$-classes of denominator m which are integrally effective on x in Theorem* 12, *are different for incongruent values x_0, and the same for congruent values x_0* (mod m).

THEOREM 13. *Let x be pure and proper* (mod m), $Nx = n$. *The number of sets $\mathfrak{L}$ of denominator m which are integrally effective on x, is equal to the number of solutions x_0* (mod m) *of*

$$x_0^2 \equiv -n \ (\text{mod } m). \tag{40}$$

For $x_0 + x$ has an unique set $\mathfrak{Q}$ of right divisors of norm m.

The number depends only on n and m, not on the particular proper x. If $m = p^r$, $(-n \mid p) = 1$, the number is 2; if $(-n \mid p) = -1$, zero.

COROLLARY 6. *Let x be a proper pure quaternion of norm n, m odd and positive. To each solution x_0 of* (40) *appertains uniquely:*

(a) *a set $\mathfrak{Q}$ of proper right divisors of norm m of $x_0 + x$;*

(b) *a set $\mathfrak{Q}$ of proper quaternions t of norm m satisfying $tx\bar{t} \equiv 0$* (mod m);

(c) *a set $\mathfrak{M}$ of pure quaternions v* (mod m) *all satisfying* (38);

(d) *a set $\mathfrak{L}$ of automorphs of denominator m integrally effective on x. Conversely each such set corresponds to one and only one* $x_0 \pmod{m}$. *Hence the number of such sets is in each case equal to the number of solutions of* (40).

Corollary 7. *The two sets appertaining to* x_0 *and* $-x_0 \pmod{m}$ *are in the same* $\mathfrak{C}$, $\mathfrak{C}$, $\mathfrak{C}$, $\mathfrak{A}$ *respectively if and only if* (32) *holds; hence certainly if two of* $x_1^2, x_2^2, x_3^2, 0$ *are equal.*

10. The degenerate cases, in which a set $\mathfrak{A}$ contains less than 24 sets $\mathfrak{L}$ are worth classifying. For any such case,

$$(40') \qquad v \equiv k(\pm i_1 v_\alpha \pm i_2 v_\beta \pm i_3 v_\gamma) \pmod{m}$$

for a choice of signs, permutation α, β, γ of 1, 2, 3, and k prime to m.

If $v \equiv k(-v_1, v_2, v_3)$ then since gcd $(v_2, v_3, m) = 1$, $k \equiv 1$, $v_1 \equiv 0$. If $v \equiv k(v_1, v_3, -v_2)$ then $v_2 \equiv -k^2 v_2$, $v_3 \equiv -k^2 v_3$, $k^2 + 1 \equiv 0$, $k - 1$ prime to m, $v_1 \equiv 0$. Similarly in all cases σ_α, $\sigma_{\alpha+1}\pi_{\alpha+1,\alpha+2}$, $\sigma_{\alpha+2}\pi_{\alpha+1,\alpha+2}$ in (21), $m \mid v_\alpha$. The only possible corresponding column of mA is by (3_1), $(\pm m, 0, 0)$. By (3) an automorph

$$(41) \qquad \begin{pmatrix} m & 0 & 0 \\ 0 & e & f \\ 0 & -f & e \end{pmatrix} \bigg/ m$$

is contained in $\mathfrak{A}$. If e is odd comparison with (10) gives $m = t_0^2 + t_1^2$, $e = t_0^2 - t_1^2$, $f = 2t_0 t_1$ in coprime integers t_0, t_1.

If $v \equiv k(v_1, v_3, v_2)$, $v_2 \equiv k^2 v_2$, $v_3 \equiv k^2 v_3$, $k^2 \equiv 1$; $k + 1$ is prime to m, for else a prime p would divide m and v_1, $v_2^2 + v_3^2 = Nv - v_1^2$, $2v_2^2$, v_2, v_3; $k \equiv 1$, $v_2 \equiv v_3$. Similarly in all cases $\pi_{\alpha+1,\alpha+2}$, $\sigma_\alpha \pi_{\alpha+1,\alpha+2}$, in (21), $v_{\alpha+1} \equiv \pm v_{\alpha+2}$. We can take the first two columns congruent (mod m), $a_{11} = e$ and $a_{22} = g$ odd and positive, the remaining $a_{\alpha 1}$ and $a_{\alpha 2}$ even. If $a_{31} = a_{32} \pm 2m$, $a_{32} = \mp m$, and we have (41). Hence $a_{31} = a_{32}$, $a_{12} = e - m$, $a_{21} = g - m$; $e^2 + a_{21}^2 = a_{12}^2 + g^2$ by (3_1), $g = e$. The two columns are $(e, e - m, f)$, $(e - m, e, f)$, where by (3_2), $f^2 = 2e(m - e)$. The third column is determined by cofactors as in (4), and we find (42), where $m = t_0^2 + 2t_1^2$, $e = t_0^2$, $f = 2t_0 t_1$:

$$(42) \qquad \begin{pmatrix} e & e - m & f \\ e - m & e & f \\ f & f & m - 2e \end{pmatrix} \bigg/ m, \qquad m^2 = (2e - m)^2 + 2f^2.$$

If $v \equiv k(v_2, v_3, v_1)$, $k^3 \equiv 1$; if $v \equiv k(-v_2, v_3, v_1)$, $k^3 \equiv -1$. In either case each v_α is prime to m, $\sum v_\alpha^2 \equiv v_1^2(1 + k^2 + k^4)$, $1 + k^2 \pm k \equiv 0$, $v_1 + v_2 \pm v_3 \equiv 0$. Thus in the last eight cases (21), $\mathfrak{A}$ contains an $(a_{\alpha\beta})$ with $a_{\alpha 1} + a_{\alpha 2} + a_{\alpha 3} \equiv 0$. By the parities, $a_{\alpha 1} + a_{\alpha 2} + a_{\alpha 3} = \pm m$, whence as $\sum a_{\alpha\beta}^2 = m^2$, $a_{\alpha 1}a_{\alpha 2} + a_{\alpha 2}a_{\alpha 3} + a_{\alpha 3}a_{\alpha 1} = 0$. If (e, f, g) and (q, r, s) are two rows, an easy elimination from

$ef + fg + ge = 0 = eq + fr + gs$ and $qe + rf + sg = qr + rs + sq$ yields $q/e = r/f = s/g$; which leads to (43) with $m = t_0^2 + 3t_1^2$, $e = t_0^2 - t_1^2$, etc.:

$$\begin{pmatrix} e & f & g \\ g & e & f \\ f & g & e \end{pmatrix} \Bigg| \; m, \qquad e + f + g = m, \qquad e^2 + f^2 + g^2 = m^2. \tag{43}$$

The case $m = 3$ belongs to both the types (42) and (43):

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \Bigg| \; 3. \tag{44}$$

THEOREM 14. *The automorph sets characterized by* (41)–(43) *are the only ones in which a set* $\mathfrak{A}$ *contains less than the maximum number,* 24, *of sets* $\mathfrak{L}$; *they are also the only ones corresponding to sets* $\mathfrak{E}$ *in which* (*cf.* (20))

$$\text{two equalities occur among } t_0^2,\ t_1^2,\ t_2^2,\ t_3^2,\ 0; \tag{45}$$

also the only ones in which two rows of $(a_{\alpha\beta})$ *form, apart from shuffling of the* x_α, *the same solution of* (5).

To prove the last part observe that distinct rows of $(a_{\alpha\beta})$ cannot have the same divisor. Hence if two rows become identical after shuffling, their divisors are 1, and as is evident from (24), (40′) holds non-trivially.

The number of sets $\mathfrak{L}$ contained in an $\mathfrak{A}$ is easily verified to be

$$4 \text{ for } (44), \qquad 6 \text{ for } (41); \qquad \text{if } m > 3,\ 12 \text{ for } (42),\ 8 \text{ for } (43). \tag{46}$$

The same proportions hold for sets $\mathfrak{Q}$ in an $\mathfrak{E}$, and sets $\mathfrak{M}$ in a $\mathfrak{C}$.

11. Let m be prime to the square part of n, x_0 a solution of (40). The form $\varphi = [m, 2x_0, l]$ of determinant $-n$, is primitive. A certain completeness is obtained in treating simultaneously automorphs of denominator m appertaining to x_0 or $-x_0$. As in §6Q every $[x]$ of norm n is carried by φ into a certain $[y]$, and by $\varphi' = [m, -2x_0, l]$ into a certain $[z]$. Here $x_0 + x = ut$, $y = (tx\bar{t})/m = tu - x_0$; $-x_0 + x = vw$, $z = wv + x_0$; $Nt = Nw = m$. Similarly, $[y]$ and $[z]$ are each carried by φ and φ' into $[x]$ and one other set [ ] not necessarily new. This chain of transformations eventually closes, and if it does not exhaust the pure quaternions of norm n, we can start a new chain with any x not already included.

If x' is obtained from x by interchanges and sign-changes of the x_α, then according as the number of these changes is even or odd, x' is carried into the similarly formed $[y']$ and $[z']$ by φ and φ', or φ' and φ; (cf. (16Q)). Thus an entire set $\mathfrak{K} = \mathfrak{K}(x)$ is carried by odd automorphs in two sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ appertaining to x_0 and $-x_0$, into two entire sets $\mathfrak{K}_1 = \mathfrak{K}(y)$ and $\mathfrak{K}_2 = \mathfrak{K}(z)$. Here $\mathfrak{A}_1 =$

$\mathfrak{A}_2$ if (32) holds. Evidently if $\mathfrak{K}$ is of weight 2, either $\mathfrak{K}_1 \neq \mathfrak{K}_2$, or $\mathfrak{K}_1 = \mathfrak{K}_2$ and is also of weight 2.

Sets $\mathfrak{K}$ of weights $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ are carried into themselves. For example, Ax is integral for $x = (g, g, 0)$ only if integral for $(1, 1, 0)$.

If $x_1 = 0$, then $x = i_1 x i_1$; $-x_0 + x = i_1(x_0 + x)i_1 = (i_1 u)(ti_1)$, $(ti_1)(i_1 u) + x_0 = -(tu - x_0) = -y$; $\mathfrak{K}_1 = \mathfrak{K}_2$. If also $m \mid x_0$, t and ti_1 are in the same $\mathfrak{O}$, $\mathfrak{A}(t)$ is of type (41) and carries x into $(0, y_2, y_3)$ of the same type.

If $x_2 = x_3$, then $t' = t_0 - i_1 t_1 - i_2 t_3 - i_3 t_2$ is a right divisor of $-x_0 + x$, $\mathfrak{A}(t')$ differs from $\mathfrak{A}(t)$ mainly in having the last two columns interchanged, and again $\mathfrak{K}_1 = \mathfrak{K}_2$. If also $m \mid x_0$, t and t' are left-associates, $\mathfrak{A}(t)$ is of type (42) and carries x into a vector $(y_1, y_2, \pm y_2)$ of the same type.

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k-SET GROUPS

BY DANIEL T. SIGLEY

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1. Introduction. The distribution of the non-invariant operators of a finite abstract group into conjugate sets has been studied by Burnside,[1] Miller,[2] and the author.[3] Some significant results which have been obtained are: The order of a finite abstract group which contains precisely k, $k > 1$, complete sets of non-invariant conjugate operators has an upper bound depending on k and, therefore, the number of such groups for a particular value of k has a finite upper bound.[4] All groups for which $0 < k \leqq 6$ are known.[5] There is no group for $k = 1$.

Corresponding investigations for non-invariant conjugate subgroups have not received explicit attention, although the determination of all the permutation groups on n letters is equivalent to the study of the distribution of the subgroups of the symmetric group of degree n into conjugate sets. The Hamiltonian groups, composed of all non-abelian groups in which every subgroup is invariant, have been studied by many authors. Dedekind[6] was the first to give a complete characterization of them.

The object of this paper is to study the finite abstract groups which contain precisely k complete sets of non-invariant conjugate subgroups. Such groups will be designated by the term *k-set* groups. The k-set groups are considered to be generalizations of the Hamiltonian groups. The 1-set and 2-set groups are determined.

2. Existence of conjugate sets of non-invariant subgroups. It is well known that the number of complete sets of conjugate operators in the direct product of two groups H_1 and H_2 which contain N_1 and N_2 such sets, respectively, and which have no operators in common except the identity, is N_1N_2. In this theorem the complete sets of conjugate operators include each invariant operator of a complete set of conjugates.

We shall state two auxilliary results concerning the existence of conjugate sets of non-invariant subgroups in the form of lemmas. The proof of these

[1] Burnside, W., *Theory of Groups of Finite Order*, 2nd ed., (1911), Note A.

[2] Archiv der Math. und Phys., vol. 17 (1910), p. 199. Trans. Amer. Math. Soc., vol. 20 (1919), p. 262. Amer. Journal of Math., vol. 54 (1932), p. 110.

[3] Duke Math. Journal, vol. 1(1935), p. 477.

[4] Loc. cit., footnotes 1, 2.

[5] Loc. cit., footnote 2.

[6] Miller, Blichfeldt, and Dickson, *Theory and Applications of Finite Groups*, (1916), p. 116.

lemmas depend upon the theorem that a necessary and sufficient condition that two non-commutative operators generate a group which contains at least one complete set of conjugate non-invariant subgroups is that they do not generate a Hamiltonian group.

Lemma 1. *Let H be any finite abstract group, and let S be any operator of finite order which transforms H into itself. Let the order of S be prime to the order of H, and let S be non-commutative with at least one operator of H. Then S and H generate a finite abstract group which contains at least one compoete set of non-invariant conjugate subgroups.*

Lemma 2. *Let H_i, $i = 1, 2, \cdots, m$, be a complete set of non-invariant conjugate subgroups in an abstract group G of finite order. Let H be an invariant subgroup of G, each of whose operators is commutative with each of the operators of H_i for $i = 1, 2, \cdots, m$. Let the order of H be relatively prime to the order of H_i. Then the subgroups of G which are the direct product of H and H_i are non-invariant and conjugate under G.*

3. On the order of a k-set group. In general, the order of a k-set group has no finite upper limit which depends on k, as may be illustrated by the fact that the non-abelian groups of order pq, where p and q are distinct prime numbers, are 1-set groups. We can, however, state a theorem which gives some information concerning the order of a k-set group.

Theorem 1. *The order of a k-set group, $k > 0$, is divisible by less than $k + 2$ distinct prime factors.*

For $k = 0$, G is Hamiltonian. If G is a simple group, the theorem is true. If G is composite, assume that the order g of G is divisible by as many as $k + 2$ distinct prime numbers. Either (i) every Sylow subgroup of G is invariant, or (ii) G contains at least one complete set of non-invariant conjugate Sylow subgroups. Under (i) G is the direct product of its Sylow subgroups and at least one of these Sylow subgroups is non-abelian and non-Hamiltonian. From Lemma 2, §2, G contains at least $k + 1$ complete sets of conjugate subgroups corresponding to the $k + 1$ direct products of the invariant Sylow subgroups and a Sylow subgroup which contains a set of non-invariant conjugate subgroups. Hence, G contains at least $k + 1$ complete sets of non-invariant conjugate subgroups.

Under (ii), let G contain x, $1 \leqq x \leqq k + 1$, invariant Sylow subgroups, and let the Sylow subgroups of the remaining $k + 2 - x$ orders form $k + 2 - x$ complete sets of non-invariant conjugate subgroups. With the use of Lemmas 1 and 2 of §2 it follows that G contains at least $N = (x + 1)(k + 2 - x)$ complete sets of non-invariant conjugate subgroups. Since every finite abstract group whose order is divisible by as many as $k + 2$ distinct prime factors involves at least $k + 1$ complete sets of non-invariant conjugate subgroups, the theorem follows.

For $k = 1$, the order of G is divisible by the maximum number of distinct prime factors that is allowed by the theorem in the non-abelian groups of order pq, where p and q are distinct prime numbers. It appears that a better lower

limit for the number of distinct prime factors which divides g may be obtained for larger values of k.

4. Existence theorem. THEOREM 2. *There exists at least one k-set group for every value of k.*

The theorem will be proved by showing how to construct such a group. First, if there exists a cyclic group which contains precisely k proper subgroups, then there exists at least one k-set group. For, if $k = 0$, this proposition is true. If $k > 0$, let n be any number of the form $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots$. Select a prime p from among the terms of the arithmetic progression $ni + 1, i = 1, 2, \cdots$ and let n contain precisely k proper divisors (excluding 1 but not n). The group G with defining relations

$$(1) \qquad \begin{aligned} &T^p = S^n = 1, \quad S^{-1}TS = T^{j \neq 1}, \quad \text{where } j^n \equiv 1, \text{ mod } p, \\ &j^m \not\equiv 1, \text{ mod } p \text{ when } n > m \geqq 1, \end{aligned}$$

is a k-set group. Now the number of divisors of n (excluding 1) is given by the formula

$$(2) \qquad \mu(n) = (\alpha + 1) \prod_i (\alpha_i + 1) - 1.$$

Thus α and α_i can be chosen so that the number n contains exactly k divisors (including n but not 1).

5. 1-set groups. Every k-set group for $k < 4$ is solvable. For if $k < 4$ and G is insolvable, G contains a simple group as an invariant subgroup or an invariant subgroup H such that the quotient group G/H is isomorphic to a simple group. Since every group whose order is divisible by less than three distinct prime factors is solvable, the simple group contains Sylow subgroups of three different orders none of which are invariant. The Sylow subgroups of each of these orders is cyclic and of prime order, otherwise an additional set of non-invariant conjugate subgroups or an invariant subgroup would result. But we have arrived at the contradiction that the order of a simple group is not divisible by the square of a prime number.

By theorem 1, the order of a 1-set group G is of the form $p^a q^b$ or p^a, where p and q are distinct prime numbers. If the order g of G is of the form $p^a q^b$, either (i) the Sylow subgroups of each of the orders p^a and q^b are invariant, or (ii) G contains one conjugate set of non-invariant Sylow subgroups, say, of order q^b. Under the first case, G is the direct product of its Sylow subgroups. That there is no 1-set group in this case follows from lemma 2.

In the second case all subgroups of G except those of order q^b are invariant. The non-invariant conjugate subgroups are cyclic. If p is odd, the Sylow subgroup of order p^a is abelian[6] and cyclic. An operator of order prime to p from the group of isomorphisms of the cyclic group of order p^a is not commutative with any operator of the cyclic group of order p^a, except the identity.

Hence $a = 1$. There is a single 1-set group of order pq^b, where p and q are distinct prime numbers with p odd. It has defining relations:

$$(3) \qquad S^p = T^{q^b} = 1, \qquad T^{-1}ST = S^\delta, \qquad \delta \not\equiv 1, \qquad \delta^q \equiv 1, \mod p.$$

If p is even, the Sylow subgroup of even order is abelian or Hamiltonian. In either case no 1-set group exists. This completes the proof that the only 1-set group whose order is divisible by two distinct prime numbers has the defining relations (3).

The only 1-set group of order p^3 is that of odd order which contains operators of order p^2 and is non-abelian. Assume that the order of G is divisible by the fourth power of the prime number p. G contains an invariant cyclic subgroup of index p. The commutator subgroup is of order p and the central is of index p^2 under G. The only 1-set group of order p^m, $m > 3$, exists for p even or odd and has defining relations:

$$(4) \qquad S^{p^a} = T^p = 1, \qquad T^{-1}ST = S^{p^{a-1}+1}, \qquad a > 2.$$

The complete sets of non-invariant conjugate subgroups in the 1-set groups of prime power order contain subgroups of order p. The results of this section may be summarized in the following theorem.

THEOREM 3. *The only* 1-*set groups are the following*: (A) $S^p = T^{q^b} = 1$, $T^{-1}ST = S^\delta$, $\delta \not\equiv 1$, $\delta^q \equiv 1$, mod *p*, *p and q prime numbers with p odd*; (B) $S^{p^a} = T^p = 1$, $T^{-1}ST = S^{p^{a-1}+1}$, $a > 2$; *and* (C) *the non-abelian group of odd prime power order* p^3 *which contains operators of order* p^2.

There are 194 0-set groups of order n (including 1) for $n < 100$. Of these 13 are non-abelian and hence Hamiltonian. There are 41 1-set set group of order less than 100, of which 7 are of prime power order.

6. 2-set groups. The results of the determination of the 2-set groups may be summarized in the following theorem.

THEOREM 4. *The only finite abstract* 2-*set groups are the following*: (A) *the direct product of a* 1-*set group G of order g and a cyclic group of prime order r, where r is prime to g*; (B) *the group G generated by extending an abelian group of order* p^2, *type* 1^2, *by an operator of prime order q which transforms the subgroup of order* p^2 *into itself and is commutative with only the identity*; (C) *the group G obtained by extending a cyclic group H of prime order p by an operator of prime power order* q^a, *where q is odd or even and not equal to p, and which transforms H according to an isomorphism of order* q^2, ($p - 1$ *divisible by* q^2); (D) *the dicyclic group of order* 16; *and* (E) *a group of order* 2^{a+2} *with defining relations* $S^{2^a} = T^4 = 1$, $T^{-1}ST = S^{-1}$, $a \geqq 2$.

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LATTICES WITH UNIQUE IRREDUCIBLE DECOMPOSITIONS

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Consider a lattice $\mathfrak{S}$ in which the ascending chain condition holds. Then each element of $\mathfrak{S}$ has at least one reduced[1] representation as a cross-cut of irreducibles. Now it is well known that the requirement that this representation be unique considerably restricts the structure of the lattice. For example, Garrett Birkhoff [1] has proved that a modular lattice in which every element is uniquely expressible as a reduced cross-cut of irreducibles is distributive. Furthermore, Morgan Ward has shown that unicity of the irreducible decompositions implies that the lattice is a Birkhoff lattice.[2] These results suggest the interesting problem of characterizing a lattice in which every element has a unique reduced representation as a cross-cut of irreducibles in terms of the structure of the lattice. We give here a complete solution of this problem. We show, namely, that such lattices are simply those Birkhoff lattices in which every modular sublattice is distributive. The detailed statement of our theorem is as follows:

THEOREM 1.1. *Let $\mathfrak{S}$ be a lattice with unit element in which every quotient lattice is of finite dimensions. Then each element a of $\mathfrak{S}$ is uniquely expressible as a cross-cut of irreducibles if and only if $\mathfrak{S}$ is a Birkhoff lattice in which every modular sublattice is distributive. In this case, for each element covering a there is exactly one irreducible not dividing it but dividing the remaining elements covering a. These irreducibles are the components of a.*

1. Throughout the paper unless otherwise stated $\mathfrak{S}$ will denote a lattice of elements $a, b, c, \cdots$ and unit element i in which every quotient lattice[3] is of finite dimensions. a covers b will be written $a > b$. An element a is said to be *irreducible* if $a = [b, c]$ implies $a = b$ or $a = c$.

DEFINITION 1.1. $\mathfrak{S}$ is said to have *unique irreducible decompositions* if every element of $\mathfrak{S}$ has a unique reduced representation as a cross-cut of irreducibles.

DEFINITION 1.2. $\mathfrak{S}$ is said to be a Birkhoff lattice if $a > [a, b]$ implies that $(a, b) > b$.[4]

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[1] A cross-cut representation is said to be "reduced" if no member of the representation is superfluous.

[2] This result was mentioned to the author in conversation. Birkhoff lattices are defined in definition 1.2.

[3] If $a \supset b$, the quotient lattice associated with a and b is the sublattice of elements x such that $a \supset x \supset b$ (Ore [1]).

[4] Cf. Klein (1).

If $\mathfrak{S}$ is a Birkhoff lattice, then there exists a rank function $\rho(a)$ (Birkhoff [2]) with the properties

(i) $\rho(i) = 0$.

(ii) $a > b$ implies $\rho b = \rho a + 1$.

(iii) $\rho[a, b] + \rho(a, b) \geqq \rho a + \rho b$.

DEFINITION 1.3. A set S of elements of $\mathfrak{S}$ is said to be *cross-cut independent* if the cross-cut of any finite number of elements of S is not divisible by any of the remaining elements of S. Similarly S is said to be union independent if the union of any finite set of elements of S divides none of the remaining elements of S.[5]

Our proof of theorem 1.1 rests on a series of lemmas. The first lemma proves the necessity while the remaining lemmas are devoted to proving the sufficiency.

LEMMA 1.1. *If $\mathfrak{S}$ has unique irreducible decompositions, then $\mathfrak{S}$ is a Birkhoff lattice in which every modular sublattice is distributive.*

PROOF. Let $a > [a, b]$ and suppose that $(a, b) \supset b_1 \supset b$ where $(a, b) \neq b_1$, and $b_1 \neq b$. Furthermore let $a = [q_1, \cdots, q_k]$ be the irreducible decomposition of a. Then since $b \not\supset b_1$ there is an irreducible component q of b such that $q \not\supset b_1$ and $q \not\supset a$. Hence $[a, b] = [q_{i_1}, \cdots, q_{i_l}, q]$ $l \leqq k$ is an irreducible decomposition of $[a, b]$. Similarly let q' be an irreducible component of b_1 such that $q' \not\supset a$. Then $[a, b] = [q_{j_1}, \cdots, q_{j_m}, q']$ $m \leqq k$ is an irreducible decomposition of $[a, b]$. But $q \neq q'$ and $q \neq q_i$ $(i = 1, \cdots, k)$. Thus $[a, b]$ has two irreducible decompositions which contradicts our assumption.

Now if there exists a modular sublattice which is not distributive, then $\mathfrak{S}$ contains a sublattice $\{u, a, b, c, d\}$ where $(a, b) = (b, c) = (a, c) = u$ and $[a, b] = [b, c] = [a, c] = d$ (Birkhoff [2]). Let $b = [q_1, \cdots, q_k]$ be the irreducible decomposition of b. Furthermore let $p_1, \cdots, p_n$ be the irreducible components of a which do not divide b and let $p'_1, \cdots, p'_m$ be the irreducible components of c which do not divide b. Then $d = [a, b] = [q_1, \cdots, q_k, p_1, \cdots, p_n]$ and dropping superfluous elements we get an irreducible decomposition $d = [q_{i_1}, \cdots, q_{i_s}, p_{i_1}, \cdots, p_{i_t}]$. In a similar manner we get an irreducible decomposition $d = [q_{j_1}, \cdots, q_{j_u}, p'_{j_1}, \cdots, p'_{j_v}]$. But $p_{i_1} \neq p'_{j_r}$ $(r = 1, \cdots, v)$ since otherwise we would have $p_{i_1} \supset (a, c) \supset b$. Similarly $p_{i_1} \neq q_{j_r}$ $(r = 1, \cdots, u)$. Hence d has two irreducible decompositions contrary to the hypothesis of the lemma.

Now let $\mathfrak{S}$ be a Birkhoff lattice in which every modular sublattice is distributive.

LEMMA 1.2. $a_1, a_2, \cdots, a_k > a$ *and* $(a_1, \cdots, a_k) \supset x \supset a$, $x \neq a$ *imply* $x \supset a_i$ *for some* i.

PROOF. We prove the lemma by induction and assume that the lemma is true for any a and $k - 1$ covering elements. Now $(a_1, \cdots, a_k) \supset (x, a_1) \supset a_1$. If $(x, a_1) = a_1$, then $a_1 \supset x \supset a$ and $x = a_1$ since $a_1 > a$. Hence $x \supset a_1$ and the lemma holds. We may thus assume that $(x, a_1) \neq a_1$. But then since $[a_1, a_2] = \cdots = [a_1, a_k] = a$ we have $(a_1, a_2), \cdots, (a_1, a_k) > a$, by definition

[5] Cf. MacLane (1), Whitney (1).

1.2. Hence by the induction assumption $(x, a_1) \supset (a_1, a_2)$ say. But then $(a_1, \cdots, a_k) \supset (x, a_1) \supset (a_1, a_2)$. If $(x, a_1) = (a_1, a_2)$ then $(a_1, a_2) \supset x \supset a$ and $x \supset a_1$ or $x \supset a_2$ by our induction assumption. We may thus assume that $(x, a_1) \neq (a_1, a_2)$. Now $(a_1, a_2, a_3), \cdots, (a_1, a_2, a_k) > (a_1, a_2)$. For $(a_1, a_2, a_i) > (a_1, a_2)$ or $(a_1, a_2, a_i) = (a_1, a_2)$ by the Birkhoff condition. But if $(a_1, a_2, a_i) = (a_1, a_2)$, then $(a_1, a_2) \supset a_i \supset a$, $a_i \neq a$ and $a_i \supset a_1$ or $a_i \supset a_2$ by the induction assumption. But this is impossible since $a_1, \cdots, a_k$ are distinct elements covering a. Now since $(a_1, \cdots, a_k) = ((a_1, a_2, a_3), \cdots, (a_1, a_2, a_k))$ we have $(x, a_1) \supset (a_1, a_2, a_3)$ say by assumption. Continuing in this manner we find that either the lemma is true or $(x, a_1) \supset (a_1, a_2, \cdots, a_{k-1})$. Hence $(a_1, \cdots, a_k) \supset (x, a_1) \supset (a_1, a_2, \cdots, a_{k-1})$ and $(a_1, \cdots, a_k) > (a_1, a_2, \cdots, a_{k-1})$. If $(x, a_1) = (a_1, \cdots, a_{k-1})$, then $(a_1, \cdots, a_{k-1}) \supset x \supset a$ and $x \supset a_j$ for some j by the induction assumption. Hence we may assume that $(x, a_1) = (a_1, \cdots, a_k) = u$. Let $a_1' = (a_2, \cdots, a_k)$. Clearly $(x, a_1') = u$ since otherwise $a_1' \supset x$ and $x \supset a_j$ by the induction assumption. Now $a_1' \supset [a_1', x] \supset a$. Hence if $[a_1', x] \neq a$, then $x \supset a_j$ by assumption. We may thus assume that $[a_1', x] = a$. Similarly we find $[a_1, x] = a$ and $[a_1, a_1'] = a$. But then $\{u, a_1, a_1', x, a\}$ is a modular, non-distributive sublattice which contradicts our hypothesis on $\mathfrak{S}$. Since the lemma is obviously true for $k = 1$, the proof is complete.

Corollary 1.1. *The elements covering any given element of $\mathfrak{S}$ are union independent.*

Corollary 1.2. *There are only a finite number of elements covering any given element of $\mathfrak{S}$.*

Lemma 1.3. *The elements covering any element a of $\mathfrak{S}$ generate a finite Boolean algebra which is dense in $\mathfrak{S}$.*

Proof. Let $a_1, \cdots, a_n$ denote the elements of $\mathfrak{S}$ covering a. Let $\mathfrak{B}$ be the sublattice generated by $a_1, \cdots, a_n$. Let $(a_1, \cdots, a_n) \supset x \supset a$, $x \supset a_1, \cdots, a_k$, and $x \not\supset a_{k+1}, \cdots, a_n$. Then $(a_1, \cdots, a_n) \supset x \supset (a_1, \cdots, a_k)$. Now $(a_1, \cdots, a_k, a_{k+1}), \cdots, (a_1, \cdots, a_k, a_n) > (a_1, \cdots, a_k)$ by definition 1.2 and corollary 1.1. Then if $x \neq (a_1, \cdots, a_k)$ we have $x \supset (a_1, \cdots, a_k, a_j) \supset a_j$ $k + 1 \leqq j \leqq n$ by lemma 1.2. This contradicts $x \supset a_j$ $k + 1 \leqq j \leqq n$. Hence $x = (a_1, \cdots, a_k)$. In particular, each element of $\mathfrak{B}$ is the union of the a_i which it divides. Set up the correspondence $x \leftrightarrow S_x$ where S_x is the set of elements a_i divisible by x. This correspondence is clearly $1 - 1$ by corollary 1.1. Furthermore $S_{(x,y)} = S_x + S_y$ and $S_{[x,y]} = S_x \wedge S_y$. Hence $\mathfrak{B}$ is a Boolean algebra.

Lemma 1.4. *Let a Birkhoff lattice $\mathfrak{S}$ have the property that any three elements covering any given element of $\mathfrak{S}$ generate a Boolean algebra of order eight. Then $a, b > [a, b]$; $q_1, q_2 \supset a$; $q_1, q_2 \not\supset b$ where q_1 and q_2 are irreducibles, imply $q_1 = q_2$.*

Proof. Suppose that $q_1 \neq q_2$. If $[q_1, q_2] \neq a$, then $[q_1, q_2] \supset a_1 > a$. But then $b_1 = (a, b) > a$ by definition 2.1 and $q_1, q_2 \supset a_1$, $q_1, q_2 \not\supset b_1$. If $[q_1, q_2] \neq a_1$, then $[q_1, q_2] \supset a_2 > a_1$ and $b_2 = (b_1, a_1) > b_1$; $q_1, q_2 \supset a_2$; $q_1, q_2 \not\supset b_2$. Hence by the ascending chain condition we may assume that $q_1, q_2 \supset a$; $q_1, q_2 \not\supset b$; $[q_1, q_2] = a$. Let $q_1 \supset a_1 > a$, $q_2 \supset a_2 > a$. This is always possible since if

$q_1 = a$ then $q_2 \neq a$ and $q_2 \supset (a, b) \supset b$ which contradicts $q_2 \not\supset b$. Now $a_1 \neq (a, b)$ since otherwise $q_1 \supset b$. Similarly $a_2 \neq (a, b)$. Hence (a, b), a_1, a_2 are three distinct elements covering a and thus generate a Boolean algebra by hypothesis. Let $a_2' = (a_1, a_2)$, $a_3' = (a_1, (a, b))$. Then $a_2' > a_1$, $a_3' > a_1$ and a_2' and a_3' are distinct. Now $q_1 \not\supset a_3'$ since otherwise $q_1 \supset b$, and $q_1 \not\supset a_2'$ since otherwise $a = [q_1, q_2] \supset a_2$. Hence there exists an element a_1' distinct from a_2' and a_3' such that $q_1 \supset a_1' > a_1$. But then a_1, a_2, a_3 generate a Boolean algebra. Hence a_1' is reducible and as before there exist elements a_1'', a_2'', a_3'' such that $a_2'' > a_1', a_2', a_3'' > a_1', a_3'$ and $q_1 \supset a_1'' > a_1'$. But then $a \subset a_1 \subset a_1' \subset a_1'' \subset \cdots$ is an infinite ascending chain which contradicts the ascending chain condition

LEMMA 1.5. *Let $\mathfrak{S}$ satisfy the hypothesis of lemma* 1.4. *Then any set of elements covering a given element a of $\mathfrak{S}$ is union independent.*

PROOF. Let us suppose that the lemma is true for any a and any $n - 1$ elements covering a. Let $a_1, \cdots, a_n > a$ and suppose that $(a_1, \cdots, a_{n-1}) \supset a_n$. Then $(a_1, a_2), \cdots, (a_1, a_n) > a_1$ and $(a_1, a_2), \cdots, (a_1, a_n)$ are distinct since any three elements covering a generate a Boolean algebra by assumption. But then $((a_1, a_2), \cdots, (a_1, a_{n-1})) \supset (a_1, a_n)$ which contradicts the induction assumption. Hence $a_1, \cdots, a_n$ are union independent.

LEMMA 1.6. *Let $\mathfrak{S}$ satisfy the conditions of lemma* 1.4 *and let $a_1, \cdots, a_n$ be the elements covering a. Then for each a_i there is one and only one irreducible q_i such that $q_i \not\supset a_i$, $q_i \supset (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$.*

PROOF. Let $a_i' = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$. Then $a_i' \not\supset a_i$ by lemma 1.5. Hence there exists an irreducible q_i such that $q_i \not\supset a_i$, $q_i \supset a_i'$. Suppose there were a second irreducible q_i' such that $q_i' \not\supset a_i$, $q_i' \supset a_i'$. Then $a_j, a_i > [a_j, a_i]$ $j \neq i$, $q_i, q_i' \supset a_j$ and $q_i, q_i' \not\supset a_i$. Hence $q_i = q_i'$ by lemma 1.4.

We will designate the irreducibles of lemma 1.6 as the irreducibles *belonging* to a. We have then

LEMMA 1.7. *Let $\mathfrak{S}$ satisfy the conditions of lemma* 1.4. *Then each element of $\mathfrak{S}$ is uniquely expressible as a reduced cross-cut of irreducibles. These irreducibles are simply the irreducibles belonging to the element.*

PROOF. If a is irreducible the lemma is trivial. Let a be reducible and let $a = [p_1, \cdots, p_k]$ be a reduced decomposition of a into irreducible components. Now if $p_i \not\supset (a_1, \cdots, a_n)$, then $p_i \not\supset a_j$ for some j. Also $p_i \supset a_k$ for some k since $p_i \supset a$. Now let q_j be the irreducible belonging to a_j according to lemma 1.6. Then $q_j \not\supset a_j$ and $q_j \supset a_k$. Hence $p_i = q_j$ by lemma 1.4. Thus either $p_i \supset (a_1, \cdots, a_n)$ or $p_i = q_j$ for some j. Furthermore for each q_l there is a p_i such that $q_l = p_i$ since otherwise $a = [p_1, \cdots, p_k] \supset a_l$ which is impossible. Hence if $p_i \supset (a_1, \cdots, a_n)$, p_i is superfluous in the decomposition of a which is contrary to assumption. Thus with suitable numbering $p_i = q_i$.

Lemmas 1.2, 1.3, and 1.7 together give theorem 1.1.

The following corollaries are a consequence of theorem 1.1.:

COROLLARY 1.3. *The number of irreducible components of an element a of a lattice having unique irreducible decompositions is equal to the number of distinct elements covering a.*

COROLLARY 1.4. *Let both the union and cross-cut decomposition of a lattice $\mathfrak{S}$ be unique. Then $\mathfrak{S}$ is distributive.*

For both $\mathfrak{S}$ and its dual satisfy the Birkhoff condition. Hence $\mathfrak{S}$ is modular and thus distributive by theorem 1.1.

2. This section will be devoted to some applications of theorem 1.1. We first give a new characterization of finite Boolean algebras.

THEOREM 2.1. *A complemented lattice of finite dimensions is a Boolean algebra if and only if every element is uniquely expressible as a reduced cross-cut of irreducibles.*

PROOF. Let u be the union of the points of $\mathfrak{S}$. If $u' \neq z$, then $u' \supset p$ and $[u, u'] \supset p$ which is impossible. Hence $u' = z$ and $u = i$. Thus the Boolean algebra generated by the points of $\mathfrak{S}$ according to lemma 1.3 is identical with $\mathfrak{S}$.

In a paper on the algebra of lattice functions, Morgan Ward has proved that a modular, non-distributive lattice satisfying the ascending chain condition always contains a complete,[6] modular, non-distributive sublattice of order five. Theorem 1.1 gives a generalization of this result.

THEOREM 2.2. *Let $\mathfrak{S}$ be a Birkhoff lattice. Then if $\mathfrak{S}$ contains a modular non-distributive sublattice it also contains a complete, modular non-distributive sublattice of order five.*

PROOF. Since $\mathfrak{S}$ contains a modular, non-distributive sublattice, not every element of $\mathfrak{S}$ is uniquely expressible as a reduced cross-cut of irreducibles by theorem 1.1. Hence by lemma 1.7 there exist three elements covering an element a of $\mathfrak{S}$ which do not generate a Boolean algebra. These three elements must then generate a complete, modular, non-distributive sublattice of order five.

In lemma 1.3 we found that any set of elements covering a given element of $\mathfrak{S}$ generate a Boolean algebra if $\mathfrak{S}$ has unique irreducible decompositions. This result may be generalized as follows:

THEOREM 2.3. *Let $\mathfrak{S}$ be a Birkhoff lattice. Then the sublattice generated by any union independent set of elements covering $a \in \mathfrak{S}$ is a Boolean algebra.*

PROOF. Let $a_1, \cdots, a_n$ be a union independent set of elements covering a. Let U be the set of elements of $\mathfrak{S}$ which can be expressed as a union of the a_i. U is obviously closed with respect to union. Let now $a = (a_1, \cdots, a_k, a_{k+1}, \cdots, a_l)$, $b = (a_1, \cdots, a_k, a'_{k+1}, \cdots, a'_m)$ where $a_1, \cdots, a_l, a'_{k+1}, \cdots, a'_m$ are distinct. Now since $a_1, \cdots, a_n$ are independent, $r(a_1, \cdots, a_k) = k$ where $r(x) = \rho(a) - \rho(x)$. Hence $r[a, b] \leqq ra + rb - r(a, b) = k + m - (k + m - l) = l$. But $[a, b] \supset (a_1, \cdots, a_l)$. Hence $r[a, b] \geqq l$. Thus $r[a, b] = l$ and $[a, b] = (a_1, \cdots, a_l)$. Hence U is also closed with respect to cross-cut and is the sublattice generated by $a_1, \cdots, a_n$. If we set up the correspondence $x \leftrightarrow S_x$ where $x \in U$ and S_x is the set of elements a_i divisible by x, then the correspondence is $1 - 1$ and $S_{(x,y)} = S_x + S_y$, $S_{[x,y]} = S_x \wedge S_y$.

If $\mathfrak{S}$ is modular it can be shown that the Boolean algebra is dense in $\mathfrak{S}$. How-

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[6] A sublattice $\mathfrak{S}'$ of $\mathfrak{S}$ is said to be *complete* if $a > b$ in $\mathfrak{S}'$ implies $a > b$ in $\mathfrak{S}$.

ever for Birkhoff lattices the Boolean algebra is in general not dense in $\mathfrak{S}$ as is easily shown by examples.

For lattices having unique irreducible decompositions the direct product decomposition may be characterized as follows:

THEOREM 2.4. *Let $\mathfrak{S}$ be a lattice satisfying the ascending chain condition in which every element is uniquely expressible as a reduced cross-cut of irreducibles. Then $\mathfrak{S}$ is the direct product of sublattices $\mathfrak{S}_1$ and $\mathfrak{S}_2$ if and only if the irreducibles of $\mathfrak{S}$ can be separated into two disjoint subsets A and B such that the set sum of any cross-cut independent set of A and any cross-cut independent set of B is again cross-cut independent.*

PROOF. Let A and B be sets of irreducibles having the property of the theorem and let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the sublattices generated by A and B respectively. Then $x \in \mathfrak{S}$ has the representation $x = [p_1, \cdots, p_r, q_1, \cdots, q_s]$ where $p_1, \cdots, p_r \in A$ and $q_1, \cdots, q_s \in B$ and $p_1, \cdots, q_s$ are cross-cut independent. Hence $x = [a, b]$ where $a \in \mathfrak{S}_1$, $b \in \mathfrak{S}_2$. Let $x = [a', b']$ where $a' \in \mathfrak{S}_1$ and $b' \in \mathfrak{S}_2$. Let $a' = [p_1', \cdots, p_t'] \; b' = [q_1', \cdots, q_u']$ be the reduced representations of a' and b'. Then $p_i' \in A$ and $q_i' \in B$ and hence $p_1', \cdots, p_t', q_1', \cdots, q_u'$ are independent by assumption. Hence $[p_1', \cdots, q_u']$ is a reduced decomposition of $[a, b]$. Thus with suitable numbering $p_i = p_i'$, $q_j = q_j'$ and $a = a'$, $b = b'$. If $x = [a, b]$, $Y = [a_1, b_1]$, then clearly $[X, Y] = [[a, a_1], [b, b_1]]$. Also $(X, Y) = [(a, a_1), (b, b_1)]$. Hence $\mathfrak{S}$ is the direct product of $\mathfrak{S}_1$ and $\mathfrak{S}_2$.

On the other hand let $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$. Then each irreducible belongs either to $\mathfrak{S}_1$ or $\mathfrak{S}_2$. Let $p_1, \cdots, p_r$ be a cross-cut independent set of irreducibles of $\mathfrak{S}_1$ and $q_1, \cdots, q_s$ be a cross-cut independent set of irreducibles of $\mathfrak{S}_2$. Suppose that $p_1 \supset [p_2, \cdots, p_r, q_1, \cdots, q_s]$. Then $p_1 \supset [p_2, \cdots, p_r]$ contrary to the assumption that $p_1, \cdots, p_r$ are cross-cut independent. Hence $p_1, \cdots, p_r, q_1, \cdots, q_s$ are independent.

If $\mathfrak{S}$ has unique irreducible decompositions then $a \supset b$ does not necessarily imply that each component of a divides some component of b. We show that this holds if and only if $\mathfrak{S}$ is distributive.

THEOREM 2.5. *Let $\mathfrak{S}$ have unique irreducible decompositions. Then $\mathfrak{S}$ is distributive if and only if $a \supset b$ implies that each component of a divides some component of b.*

PROOF. Let $a \supset b$ imply that each component of a divides some component of b. Let $q \supset [a, b]$ where q is irreducible. Then $q \supset [p_1, \cdots, p_r, q_1, \cdots, q_s]$ where $p_1, \cdots, p_r \supset a$; $q_1, \cdots, q_s \supset b$ and $[p_1, \cdots, q_s]$ is the irreducible decomposition of $[a, b]$. Then by hypothesis $q \supset p_i$ or $q \supset q_j$ and hence $q \supset a$ or $q \supset b$. Thus every irreducible is a prime. Let $([a, b], [a, c]) = [p_1, \cdots, p_r]$. Then $p_i \supset [a, b]$, $p_i \supset [a, c]$. If $p_i \not\supset a$, then $p_i \supset b$, $p_i \supset c$. Hence either $p_i \supset a$ or $p_i \supset (b, c)$. Thus in either case $p_i \supset [a, (b, c)]$. Therefore $([a, b], [a, c]) \supset (a, [b, c])$. Hence $([a, b] [a, c]) = (a, [b, c])$ and $\mathfrak{S}$ is distributive.

Conversely if $\mathfrak{S}$ is distributive, then every irreducible is a prime (Ward [1]) and $a \supset b$ implies $p_i \supset [q_1, \cdots, q_s]$ implies $p_i \supset q_j$ for some j.

Theorem 1.1 gives finally the following result on the sublattice of a lattice having unique irreducible decompositions.

THEOREM 2.6. *Let $\mathfrak{S}$ have unique irreducible decompositions. Then a sublattice $\mathfrak{S}'$ of $\mathfrak{S}$ has unique irreducible decompositions if and only if it is a Birkhoff lattice.*

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TOPOLOGICAL TRANSFORMATION GROUPS. I

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1. A topological group G is said to be a *transformation group* of a space E if the following conditions are satisfied:

1. To every element g in G, there is associated a homeomorphism, $g(x)$ of E into itself.
2. To the identity, g_0, of G is associated the identity homeomorphism.
3. If g_1 and g_2 are any two elements of G, then

$$g_1[g_2(x)] = (g_1g_2)(x).$$

4. The function $g(x)$ is continuous simultaneously in the two variables g and x.

The group G will be assumed to be compact and metric or the equivalent of this in some other formulation.[1] We recall that G is said to be an *effective* transformation group if every element of G, with the exception of g_0, moves at least one point of E. An *orbit* of G is a set of points $G(x)$ of E; that is the set of all points $g(x)$ where x is fixed and g is allowed to vary over G. For any y in $G(x)$, it is true that $G(x) = G(y)$.

If G is not effective and if G^* is the subgroup leaving every point of E fixed, then G^* is invariant[2] and closed and G/G^* may be defined in a natural way as a transformation group of E which is effective. In case G^* is any (not necessarily the maximal) closed invariant subgroup leaving E fixed, then G/G^* may be defined as a transformation group of E though it is not necessarily effective.[3]

In studying transformation groups we are interested in relations among the properties of G, the orbits of G, and the space E. The space E will be subjected to various restrictions but for the sake of definiteness we shall always take E to be metric, although for some of our theorems this condition can be relaxed. An important tool in studying transformation groups and a concept of interest in itself is that of the *orbit space*. This is a decomposition space (Zerlegungsraum) in which the orbits $G(x)$, for x in E, are regarded as points. The topology of this space is the usual one for decomposition spaces, the orbits being compact

[1] This is a blanket assumption running through the entire paper and we do not restate it in every hypothesis. Many of the definitions used in this paper will be found in earlier papers. See (5, 6, 7) where numbers in parentheses refer, as is conventional, to the bibliography at the end of the paper.

[2] More generally the subgroup leaving any invariant set fixed is invariant. It is also closed.

[3] The method of defining G/G^* as a transformation group is as follows: Let $\bar{g}$ be any element of G/G^* and let g be any element of G which is mapped into $\bar{g}$ by the homomorphism $G \rightarrow G/G^*$. Then $\bar{g}(x)$ is defined to be $g(x)$.

subsets of the metric space E; and as we have noted elsewhere the transformation from E to its decomposition space is interior.

In the present paper we point out certain facts concerning the structure of orbits which follow from the known structure of G. It is shown that the local structure of a finite dimensional orbit is similar to the local structure of a finite dimensional compact group. A fact which is useful in many connections is that points of E which are on orbits of dimension at least k form an open set. If G acts effectively on a finite dimensional orbit then the group is finite dimensional; in other words if G is transitive and effective on a finite dimensional space, then G must be finite dimensional; in fact an upper bound for the dimension of G is obtained. If G acts effectively on a locally Euclidean space and if its orbits are locally connected then G must be a Lie group. This, it should be noted, is a theorem purely about the structure of G and not a theorem about the way G acts on the space. Some of the theorems of the paper could be more easily proved if one made, as we do not, the assumption that G is finite dimensional.

2. If x is any point in E there is a certain closed subgroup of G, possibly containing only g_0, which leaves x fixed. This subgroup, denoted by G_x, may or may not be invariant but at any rate it divides G into a continuous family of left cosets, that is sets of the type gG_x. This space is, of course, homeomorphic to the orbit $G(x)$ and hence the study of orbits is closely connected with the study of coset spaces. We will now study certain properties of coset spaces by means of the structure theorems for compact groups (3, 5, 9, 11, 12).

Let G be any compact group and let

$$\text{(A)} \qquad H_1, H_2, H_3, \cdots; \qquad H_i = f_i(H_{i+1})$$

be its defining sequence of Lie groups,[4] where the f_i's are continuous homomorphisms. Let G^* be a closed subgroup of G, not necessarily invariant. If H_i^* is the subgroup of H_i associated with G^* then H_i^*, is a closed subgroup of a Lie group, is also a Lie group (2) and $H_1^*, H_2^*, H_3^*, \cdots$ is a defining sequence of Lie groups for G^*. Consider the sequence of coset spaces (these are manifolds; see 13).

$$\text{(B)} \qquad H_1/H_1^*, H_2/H_2^*, H_3/H_3^*, \cdots$$

with a transformation $H_i/H_i^* = f_i^*(H_{i+1}/H_{i+1}^*)$ defined as follows:

$$\text{(C)} \qquad \textit{if } h_i = f_i(h_{i+1}), \textit{ then } f_i^*(h_{i+1}H_{i+1}^*) = h_iH_i^*.$$

THEOREM 1. *The functions f_i^* defined in* (C) *are continuous, uniquely defined, and the system given by* (B) *and* (C) *has a limit which is homeomorphic to the coset space G/G^*.*

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[4] Spaces, whether group spaces or not, which are defined in terms of such sequences are thoroughly treated in (4). The fact that any compact group can be represented as in (A) is proved in the references already given on structure theorems.

It will first be shown that the definition of f_i^* is unique. Let h_{i+1} and h'_{i+1} be two elements of H_{i+1} yielding the same coset, $h_{i+1}H_{i+1}^* = h'_{i+1}H_{i+1}^*$. Then there is a g_{i+1} in H_{i+1}^* such that $h_{i+1} = h'_{i+1}g_{i+1}$ so that, by the definition of a homomorphism $h_i = h'_i g_i$[5] where g_i is in H_i^*. Then $h_iH_i^* = h'_iH_i^*$ and the uniqueness of the definition has been shown.

The continuity of f_i^* follows from the fact that if $h_{i+1}H_{i+1}^*$ and $g_{i+1}H_{i+1}^*$ are cosets which are near each other then there is no loss of generality in assuming g_{i+1} and h_{i+1} to be near each other. Then g_i and h_i are near each other and their associated cosets are near.

It has been shown that (B) and (C) satisfy the usual conditions used in defining a space by this method. There is yet to be shown the fact that the space thus defined is homeomorphic to G/G^*.

Let gG^* be a coset of G/G^* where $g = (h_1, h_2, \cdots)$. Then we have a function defined on gG^* to the limit space if we associate with the above element the element

$$(h_1H_1^*, h_2H_2^*, \cdots). \tag{D}$$

The correspondence thus defined is single valued; it is furthermore one-to-one, because if gG^* and $g'G^*$ are distinct cosets of G/G^* the correspondence yields two distinct sequences (D).

If the cosets gG^* and $g'G^*$ are near each other g and g' may be taken near each other. This shows that the correspondence is continuous and hence topological.

LEMMA 1. *If Q is an invariant subgroup of G then for any x and y in the same orbit $Q(x)$ and $Q(y)$ are homeomorphic.*

Let $y = g(x)$. Then $Q(y) = Qg(x) = gQ(x)$, that is $Q(y)$ is obtained by operating on $Q(x)$ with g.

In particular we see that if $Q(x)$ is a point then $Q(y)$ is a point for all points y in the orbit of x and the group Q is *idle* (*inoperative*) on the orbit. Hence if a group is effective on an orbit no invariant subgroup can have a fixed point.

The word discontinuum is used throughout this paper as meaning a compact zero dimensional set; in particular cases, then, such a set may be finite or a single point.

3. THEOREM 2. *Let G be an n-dimensional connected group effective on an orbit $M = G(x)$. Then M is locally the topological product of a k-cell, $1 \leqq k \leqq n$, by a discontinuum.*

Let Z be the central subgroup of G such that G/Z is an n-dimensional Lie group and such that G is locally the direct group product of Z by a local Lie group L. Let G^* denote the set of elements g of G such that $g(x) = z(x)$ for an appropriate choice of z in Z. The set $G^*(x) \subset Z(x)$; these sets are in fact iden-

[5] The subscript i is used to convey that these elements are in H_i, that is for example $h_i = f_i(h_{i+1})$.

tical. Since the group Z is central every subgroup is invariant and $z(x) = x$ for the identity only because of the fact that G is effective.

We will show now that G^* is a group. Let g_1 and g_2 belong to G^* so that for some choice of z_1 and z_2, $g_1(x) = z_1(x)$ and $g_2(x) = z_2(x)$. But then $g_1g_2(x) = g_1z_2(x) = z_2g_1(x) = z_2z_1(x) = z(x)$ and g_1g_2 belongs to G^*.

If $g(x) = z(x)$ it follows that $x = g^{-1}z(x) = zg^{-1}(x)$ and hence $z^{-1}(x) = g^{-1}(x)$ and g^{-1} belongs to G^*. These two facts show that G^* is a subgroup of G and it should be noted that it is a closed subgroup.

The group G^* contains G_x and Z. Moreover for g in G^*, $g(x) = z(x)$ and $x = z^{-1}g(x)$. Therefore $z^{-1}g$ is in G_x. Since $g = zz^{-1}g$ it follows that if g is in G^* it may be represented as the product of two elements one in Z and one in G_x. Furthermore G_x and Z have in common only the identity for this is the only element of Z leaving x fixed. Then G^* is the direct product of G_x and Z, for these groups clearly commute.

Since Z is a discontinuum it follows that the component of G^* is K_x, the component of G_x.

As we have said Z and L, a local Lie group, are such that their product gives us an open set which is a local subgroup of G. Let L_x denote the intersection of G_x and L. Then L_x as a closed subgroup of a local Lie group is a local Lie group (2). The local group L may be chosen as so small that every element of L_x is in the component K_x of G_x. Because $Z \times L$ is open in G and because the mapping of G on M is an interior one, it follows that $(Z \times L)(x)$ is an open subset of M.

Let L^1 be an open local subgroup of L chosen so that L^1L^1 is in L. It will now be shown that if the sets $zL^1(x)$ and $z'L^1(x)$ have a point in common then z coincides with z'. Assuming this to be false we have $zg(x) = z^1g'(x)$ for g and g' in L^1 and z and z' distinct. From this we deduce that $g'^{-1}g(x) = z^{-1}z'(x)$. The element g'^{-1} belongs to L by the choice of L^1 and it also belongs to G^* as the relation above shows. The direct product character of G^* shows that it has no elements in L except L_x. Therefore $g'^{-1}g$ is in L_x and $z^{-1}z'$ leaves x fixed from which it follows that $z^{-1}z'$ must be the identity.

This assertion shows that $(L^1 \times Z)(x)$ is the topological product of the set $L^1(x)$ and $Z(x)$. This last set is a discontinuum homeomorphic to $Z(x)$. The set $L^1(x)$ is locally euclidean because it is isomorphic to the space L^1/L^1_x where L^1_x is the local Lie subgroup of L^1 which leaves x fixed (see 13 for a discussion of a similar point). In order to obtain the desired local representation of M it is only necessary to choose a subset C of $L^1(x)$ which is an n-cell and take the product of C and $Z(x)$.

4. It is understood in the following corollaries that the hypothesis is the same as for the above theorem.

COROLLARY 1. *For any two elements z and z' of Z the sets $zL^1(x)$ and $z'L^1(x)$ have no common point.*

This was shown in the course of the above argument.

COROLLARY 2. *The group G_x is a Lie group.*

The group G/Z is a Lie group. In the mapping G_x is carried into a closed subgroup which must be a Lie group (2). This group is G_x factored with respect to the intersection of G_x and Z; but this intersection is the identity alone.

COROLLARY 3. $\dim G_x + \dim M = \dim G$.

The dimension of M is equal to the dimension of $L^1(x)$ which is equal to L^1/L^1_x. The above relation is equivalent to the following one which is not difficult to see: $\dim L^1_x + \dim L^1/L^1_x = \dim L^1$.

COROLLARY 4. *The following inequalities are true*:

$$1 \leqq \dim M \leqq n, \qquad 0 \leqq \dim G_x \leqq n - 1.$$

The second inequality follows from the first and the first follows from the fact that M is connected (since G is) and not merely a single point.

COROLLARY 5. *The group G contains a one-parameter local subgroup T which operates on $M = G(x)$ in such a way that if t and t^1 are distinct elements in T, then $t(x) \neq t^1(x)$.*

By corollary 4, L_x is of dimension at least one less than $\dim L$. Hence L must contain a one-parameter local subgroup T^* which has only the identity in common with L_x. We shall show that a suitable germ in this operates on x as required by the corollary.

If no such germ has the desired property then there is a pair of distinct elements g and g' in T^* and in any neighborhood of the identity such that $g(x) = g'(x)$. But then $g^{-1}g'(x) = x$ and $g^{-1}g'$ is near the identity. It has thus been shown that there are elements of T^* in every neighborhood of the identity leaving x fixed. The set of these elements is a closed subgroup of T^* and must coincide with T^*. But this shows that T^* is in L_x which contradicts the way T^* was chosen.

COROLLARY 6. *The orbit M is locally connected if and only if G is a Lie group.*

If G is a Lie group it is known that M is a manifold (13). On the other hand if M is locally connected Z must be finite. Since G/Z is always a Lie group, when Z is finite G is also a Lie group.

5. In the following theorem there is no restriction on the dimensionality or connectedness of G. We say that a group is effectively finite dimensional on a space if the associated effective group is finite dimensional.

THEOREM 3. *Let G be a group operating on an orbit $M = G(x)$ of dimension greater than zero. If G is effectively finite dimensional on M, then G contains a one-parameter local group which operates homeomorphically on x.*

There is no loss of generality in taking G to be connected for if it isn't we take its component which would also have an orbit of positive dimension. Let K^* be the subgroup of G leaving all of M fixed. Then by the preceding corollary G/K^* must contain a one-parameter local group T^* which operates homeomorphically on x. Let J be a one-parameter group in G covering T^* isomorphically (see Pontrjagin (2)), that is if f denotes the mapping of G onto

G/K^* then $f(j)$, j in J, is the local group T^*. We assert that J operates homeomorphically on x. For if it does not, then there is a g and a g', both in J and distinct, such that $g(x) = g'(x)$. Let $f(g) = g^*$ and $f(g') = g'^*$. The above relation implies that $g^*(x) = g'^*(x)$. But this is impossible.

LEMMA 2. *Let G be a transformation group of a finite dimensional connected orbit $M = G(x)$ and let K be an invariant connected subgroup such that G/K is finite dimensional and $K(x)$ is a proper subset of M. Then* dim $K(x)$ *is less than* dim M.

Since K is invariant it follows that G is a topological transformation group of the decomposition space of M induced by the sets $K(y)$ as y varies in M; this space will be denoted by $D(M; K)$. The above statement follows from the fact that $gK(y) = Kg(y)$. Now K is idle on $D(M; K)$ so that G/K is a finite dimensional group acting on $D(M; K)$. We may suppose dim $M > 0$, for otherwise the Lemma is vacuously true. Then $D(M; K)$ is connected and not a single point; so that dim $D(M; K) > 0$.

From the preceding theorem there is in G a one-parameter local group J which acts homeomorphically on $K(x) = x^*$. The group J also acts on M and the set $J(x)$ cannot have two points in common with a set $K(y)$; otherwise the corresponding point in $D(M; K)$ could be obtained by operating with two distinct elements of J on x^*. The sets $K(y)$ are homeomorphic since K is invariant and therefore $J[K(x)]$ is the topological product of an arc and the set $K(x)$. Hence $J[K(x)]$ is of larger dimension than $K(x)$ by a theorem due to Hurewicz (4.1). This set is a subset of M and our theorem is established.

6. THEOREM 4. *If G acts on a finite dimensional orbit $M = G(x)$ then G is effectively finite dimensional on M. In fact, there must be a connected invariant group K which is idle on M and such that G/K is finite dimensional.*

Let $G_1, G_2, \cdots$ be a sequence of invariant subgroups of G, shrinking toward the identity monotonically and such that G/G_n is a Lie group. Let K_n be the identity component of G_n. These are invariant subgroups of G, and G/K_n is finite dimensional. The groups K_n converge to the identity and we wish to show that some one of them is inoperative for in this case G is effectively finite dimensional.

It is certain at any rate that K_n lies in an arbitrarily small neighborhood of the identity for n sufficiently large and therefore for some n, say $n = N$, $K_N(x)$ is of lower dimension than $G(x)$. We may now repeat this process starting with the orbit $K_N(x)$. In order to do this it is only necessary to observe that if $n > N$, K_n is an invariant subgroup of K_N and that K_N/K_n is finite dimensional. The repetition of the process yields a K_R in a finite number of steps such that $K_R(x)$ is zero dimensional. Since K_R is connected, $K_R(x) = x$ and this means that K_R is idle on $G(x)$. The proof of the theorem is thus completed.

This enables us to extend the results about the structure of an orbit for the case where G is finite dimensional and connected to the case where G is merely assumed to be connected.

COROLLARY 7. *Let G be a connected group acting on a finite dimensional orbit $M = G(x)$. Then M is locally the topological product of a k-cell and a discontinuum.*

7. Before proceeding we prove a generalization of corollary 3 to the case where G is neither connected nor effective.

COROLLARY 3′. *Let G be a finite dimensional group acting on an orbit $M = G(x)$. Then*

$$\dim G_x + \dim M = \dim G.$$

Let G^* be the invariant subgroup leaving all of M fixed. Then it is known that $\dim G^* + \dim G/G^* = \dim G$. But $G^* \subset G_x$. Then also $\dim G^* + \dim G_x/G^* = \dim G_x$. Hence to prove the relation of the corollary we have only to prove that (since $(G/G^*)_x = G_x/G^*$) $\dim (G/G^*)_x + \dim M = \dim G/G^*$, that is it may be assumed that G is effective. We therefore make that assumption, and go on to prove the corollary in that case.

Let K be the component of G. We know from corollary 3 that $\dim K_x + \dim K(x) = \dim K$. The corollary now follows from the facts that $\dim K = \dim G$. $\dim K_x = \dim (K \cap G_x) = \dim G_x$; $\dim K(x) = \dim G(x)$.

THEOREM 5. *Let G be a group given by* (A) *and let G^* be any compact subgroup. Then the dimension of G/G^* is equal to the limit of the dimensions of the spaces in* (B), *that is*

$$\dim G/G^* = \lim \dim H_i/H_i^* .$$

If $\dim G/G^*$ is infinite then $\dim H_i/H_i^*$ must approach infinity for otherwise G/G^* would be transformed by small transformations into sets of bounded dimension and this is impossible.

Let us suppose then that G/G^* is finite dimensional. Now in this case G may be regarded as a topological transformation group of a finite dimensional orbit M. The set M is simply the space of cosets G by G^*. The theorem that G must be effectively finite dimensional in its action on M translates itself into this that there is an invariant subgroup G_0 of G^*, as well as G, such that G_0 is idle on M and G/G_0 is finite dimensional. Now the coset space G by G^* is homeomorphic to the coset space of G/G_0 by G^*/G_0. Similar considerations obtain in the approximating groups and the associated coset spaces. There is therefore no loss in assuming that the "factoring" by G_0 has been accomplished throughout: so that the group G may be thought of as finite dimensional, say n-dimensional. Let G^* be r-dimensional. From theorems on the structure of groups there must be an integer N such that if $i \geqq N$ $\dim H_i = n$ and $\dim H_i^* = r$. By corollary 3′, $\dim G/G^* = n - r$ and if $i \geqq N$ $\dim H_i/H_i^* = n - r$. Therefore $\dim G/G^* = \lim \dim H_i/H_i^*$.

8. We now prove a theorem which will enable us to draw a conclusion about the distribution of points on orbits of at least a given finite dimension. It is convenient to prove first a Lemma.

LEMMA 3. *In the sequence of coset spaces of* (B) *the dimension never decreases.*

Let

$$\dim H_i = n_i, \qquad \dim H_{i+1} = n_{i+1};$$
$$\dim H_i^* = r_i, \qquad \dim H_{i+1}^* = r_{i+1}.$$

Let $\bar{H}_{i+1}$ be the invariant subgroup of H_{i+1} mapped into the identity by f_i and let $\dim \bar{H}_{i+1} = \bar{r}$. Then

$$n_i = n_{i+1} - \bar{r},$$

$$r_i = r_{i+1} - \dim (H_{i+1}^* \cap \bar{H}_{i+1}) \geqq r_{i+1} - \bar{r}.$$

Subtracting

$$n_i - r_i \leqq n_{i+1} - r_{i+1}.$$

THEOREM 6. *Let G^n be a sequence of closed subgroups of the compact group G approaching the closed subgroup G^* (in the sense of drawing into any open set inclosing G^*). Then if* $\dim G/G^* = r$ *and n is sufficiently large* $\dim G/G^n \geqq r$.

We first make the proof under the assumption that G is a Lie group. In this case, G^n must be a Lie subgroup of G. It will be shown that for large values of n, $\dim G^n \leqq \dim G^*$ which is equivalent to the desired conclusion.

Suppose that for an infinite number of values of n, $\dim G^n > \dim G^*$. Let P^n and P^* be the tangent hyperplanes to G^n and G^*. Then P^n, for an infinite number of n's, must contain a vector V_n orthogonal to P^*. We may assume that $V_n \to V$ where V too is orthogonal to P^*. The group in G determined by V must attain a certain distance ϵ from G^*. But groups determined by V_n must remain near those determined by V. This shows that the G^n's cannot move into every open set including G^*. This contradicts our supposition and the theorem follows.

Proceeding now to the general case, let $H_1^n, H_2^n, \cdots$ be the sequence of Lie groups determining G^n; the group H_i^n is a subgroup of H_i (of the system A). By the preceding results, G/G^n is the limit of H_i/H_i^n and hence

$$\dim G/G^n = \lim \dim H_i/H_i^n.$$

In view of the special case just treated and in view of the fact that as n becomes large H_i^n must approach (in the sense specified) H_i^* it is true that for a fixed i_1 and for $n \geqq N$,

$$\dim H_{i_1}/H_{i_1}^n \geqq \dim H_{i_1}/H_{i_1}^*.$$

We assume that $\dim H_{i_1}/H_{i_1}^* = r$, so that if $n \geqq N$, $\dim H_{i_1}/H_{i_1}^n \geqq r$. Then for $n \geqq N$ and $i \geqq i_1$, $\dim H_i/H_i^n \geqq r$, and if $n \geqq N$

$$\dim G/G^n = \lim \dim H_i/H_i^n \geqq r.$$

THEOREM 7. *If G acts on a space E then the points on orbits of dimension $\geqq k$ form an open set.*

Because k is always an integer the theorem could equally well be stated in terms of a pure inequality.

Let x be any point of E. The theorem is an immediate consequence of Theorem 6 and the fact that if y is sufficiently near x then G_y is near G_x.

9. We wish next to prove a theorem giving an upper bound to the dimension of a group which can act effectively on an orbit of dimension k. A few preliminary Lemmas and a Theorem will be proved first some of which are of interest in themselves.

If M is any invariant set the subgroup G_M leaving all of M fixed is an invariant subgroup and the component K_M of G_M is invariant in G_M and it must in fact be an invariant subgroup of the group G.

LEMMA 4. *For all orbits N sufficiently near to an orbit M it is true that K_N lies in a preassigned neighborhood V of K_M.*

If this is not true there is a sequence of elements g_i belonging to K_{N_i} where the orbits N_i converge to N and g_i is not in V. The elements g_i may be assumed to converge to an element g not in V, and the sets K_{N_i} may be assumed to converge to a set F. The set F is a connected subset of G_M and hence is in K_M. The element g, which is in F, must therefore be in K_M. But since it is not in V a contradiction has been reached.

THEOREM 8. *For all orbits N sufficiently near to a finite dimensional orbit M it is true that K_N is a subgroup of K_M.*

The group G/K_M is finite dimensional and from the local structure of a compact group there is a neighborhood U of the identity of G/K_M which contains no connected subgroup with the exception of the degenerate group containing only the identity element. The inverse of U is a neighborhood V of the identity in G; the neighborhood V contains K_M in its interior. By the preceding Lemma if N is near to M, K_N must be in V. Then K_N is mapped into U which contains no connected subgroups except the identity and therefore K_N is mapped into the identity. This means that K_N is a subgroup of K_M, and the proof of the theorem is complete.

It is true that G_N draws into any given neighborhood of G_M when N approaches M but it cannot be argued in general that G_N is a subgroup of G_M. This is brought out by the following counter-example.

Let G be a finite dimensional non-Lie group defined by the sequence H_i of Lie groups. The space E is to be the totality of elements contained in G and in all the sets H_i. Let x be a point of E. If x is in an H_i the neighborhoods of x are its neighborhoods in H_i; if x is in G, k is any integer and O is a neighborhood of x in G, a neighborhood of x in O is defined as all elements of O plus all elements in H_i $(i \geqq k)$ which are "coordinates" of elements in 0.

Now G is a topological transformation of E with the orbits $H_1, H_2, \cdots$ and G. The set G_G is the identity but G_{H_i} is a non-degenerate zero dimensional group.

LEMMA 5. *If G is a Lie group which is effective on a connected locally euclidean space E, then G_M is finite for some orbit M of E.*

Let x be a point of E on an orbit $M = G(x)$ such that G/G_M is of the highest

possible dimension. Such a choice is possible because there are only a finite number of possible dimensions. For orbits N sufficiently near to M, K_N is a subgroup of K_M. The groups K_N and K_M are Lie groups and connected. If K_N is a proper subgroup of K_M it must be of lower dimension, and this would imply that G/K_N is of higher dimension than G/K_M which contradicts the choice of M. From this it follows that for all orbits N near M, K_M and K_N are identical. This means that K_M is idle in an open subset of E. Whenever a Lie group is idle on an open subset of E it must be idle on all of E.[6] But G is effective and so K_M must be the identity, that is G_M is finite.

LEMMA 6. *If G is effective on a zero-dimensional set then G must be zero-dimensional.*

The proof is omitted.

These Lemmas enable us to prove the theorems on the limitation of the dimension of effective groups. The proof is in two theorems beginning with the case of Lie groups.

10. THEOREM 9. *Let G be a Lie group effective on an orbit M whose dimension is k. Then* $\dim G \leqq \frac{1}{2}k(k+1)$.

The theorem is true for the case where $k = 0$ by the above Lemma. The proof will be by induction, and we now assume that the conclusion is true for any orbit of dimension at most $k - 1$.

Consider G_x for any point x of M. This is a Lie group acting effectively on M and therefore there is some orbit $N = G_x(y)$ such that $(G_x)_N$ is finite. Then the effective group $G_x/(G_x)_N$ is of the same dimension as G_x. By the hypothesis of the induction $\dim G_x$ is at most $\frac{1}{2}(k-1)k$ because $G_x(y)$ is at most $(k-1)$-dimensional. That $G_x(y)$ is indeed at most $(k-1)$-dimensional requires, perhaps, a word of justification. In the simpler case that M is connected, M is a manifold and $G_x(y)$ a finite set of manifolds forming a proper subset of M. In this case it is trivial that $\dim G_x(y) < \dim M$. In case M is not connected it is the sum of a finite number of manifolds and it is rather easy to show that the identity component of G, of the same dimension as G, acts effectively on one of the components of M. This reduces the argument to the earlier case. Now, $\dim G = \dim G_x + \dim M = \frac{1}{2}(k-1)k + k = \frac{1}{2}k(k+1)$.

COROLLARY 8. *If a Lie group is effective on a locally euclidean space and if the orbits have highest dimension k then*

$$\dim G \leqq \tfrac{1}{2}k(k+1).$$

This is because the effective group on at least one orbit is (by Lemma 5) of the same dimension as G.

We now prove a fact which will be of use in the next theorem. Let G be

[6] This is due to the fact that in a Lie group periodic elements are everywhere dense and to the fact that a periodic transformation of a connected locally euclidean space leaving an open set fixed must leave the entire space fixed. This theorem is due to Newman (10). For generalizations and alternative proofs see Smith (14, 15).

group given by the sequence (A) and let G^* be a subgroup determining an orbit on which G acts effectively. The fact we will need is that H_i acts effectively on H_i/H_i^*. Let f be the homomorphism taking G into H_i and let h be any element of H_i. Let g be some element in $f^{-1}(h)$. There must be a g' such that $gg'G^* \neq g'G^*$ and this implies that $g'^{-1}gg'G^* \neq G^*$. This means that $g'^{-1}gg'$ is not in G^* and therefore $f(g'^{-1}gg^1) = h'^{-1}hh'$ is not in H^*. Hence $h'^{-1}hh'H^* \neq H^*$ and $hh'H^* \neq h'H^*$, that is h moves a point of H_i/H_i^*.

THEOREM 10. *Let G be effective on an orbit M of dimension k. Then* dim G *is at most* $\frac{1}{2}k(k+1)$.

Let G be given by the sequence (A) and let G^* be the closed subgroup of G determining the orbit. Let G^* be determined by $H_1^*, H_2^*, \cdots$ so that the orbit M is determined by the sequence (B). Because M is k dimensional it is true for all i's after a certain one that dim $H_i/H_i^* = k$. From the preceding theorem and the fact that H_i must act effectively on H_i/H_i^* we conclude that dim $H_i \leqq \frac{1}{2}k(k+1)$. The fact that the above inequality holds for all i's after a certain one shows that dim $G \leqq \frac{1}{2}k(k+1)$.

11. THEOREM 11. *Let G be a group acting effectively on a connected space E. If all the orbits of E are of the same dimension k, then G must be finite dimensional.*

Let M be any orbit of E. Then K_M is such that G/K_M is finite dimensional. For N near to M, K_N is in K_M. Choose M so that G/K_M is of the highest possible dimension; now, for N near M, K_N must equal K_M. Then the set of points left fixed by K_M is open. It is always closed, and K_M must be idle on all of E which shows that it is the identity and that G is finite dimensional.

12. As a preliminary to extending Corollary 6 to the disconnected case and to showing that an effective group in a connected locally euclidean space with locally connected orbits must be a Lie group we prove the following Lemmas.

LEMMA 7. *If two spaces H_i/H_i^* and H_{i+1}/H_{i+1}^* of the sequence* (B) *are of the same dimension then f_i^* is a local homeomorphism.*

Let $h_{i+1}H_{i+1}^*$ be an element of the second space. We wish to show that no two elements near this element have the same image under f_i^*.

Let $\bar{H}_{i+1}$ be the kernel of the homomorphism f_i, that is the invariant subgroup of H_{i+1} made up of all elements which f_i takes into the identity of H_i. The group $\bar{H}_{i+1}$ may be considered as a transformation group of H_{i+1}/H_{i+1}^*. The orbits under this group are precisely the sets in H_{i+1}/H_{i+1}^* which are taken by f_i^* into single elements of H_i/H_i^*. These orbits must all be finite, for otherwise the dimension would be lowered in passing from one space to the other.

The orbits in question are given by an invariant subgroup and are homeomorphic. Since they are finite this means that they all have the same number of points. Hence no two "points" near $h_{i+1}H_{i+1}^*$ can be on the same orbit and the demonstration is complete.

LEMMA 8. *If the sequence* (B) *defines a locally connected orbit of finite dimension then there is an integer N such that if $i \geqq N$, f_i^* is a homeomorphism.*

Since the orbit is finite dimensional there is an N such that if $i \geqq N$, all the

approximating coset spaces are of the same dimension. For such values of i, f_i^* is a *local* homeomorphism. Assume that for an infinite set of i's f_i^* is not a homeomorphism. Now drop from the sequence

$$(1) \qquad H_1/H_1^*, H_2/H_2^*, \cdots$$

all the spaces preceding the N^{th} and drop all those which go by a homeomorphism into the preceding space. Assume now that (1) is the resulting sequence of spaces, which will of course have the same limit space as before. And now each f_i^* is a local homeomorphism but not a homeomorphism.

By the theory of covering spaces if O is any open set in H_i/H_i^* homeomorphic to the interior of a k cell (we assume H_i/H_i^* to be k-dimensional) and if r points of H_{i+1}/H_{i+1}^* cover each point of H_i/H_i^* then there are precisely r open mutually disjoined subsets of H_{i+1}/H_{i+1}^* which are taken homeomorphically into O.

We proceed now analogously to the argument for groups. Let O_1 be an open k-cell in H_1/H_1^*, and let $O_1^2, \cdots, O_2^{r_2}$ be the disjoined open sets in H_2/H_2^* going homeomorphically into O_1. We may continue to the third space in the same way and so on indefinitely. This shows that the limit space is in the small the topological product of a k-cell and an infinite compact zero dimensional set. One of the sheets of the product for example is given by $O_1, O_2^1, \cdots$. Such a space fails to be locally connected. We have now been led to an absurdity in assuming that there were an infinite number of i's in the original sequence for which f_i^* was not a homeomorphism, and the conclusion follows. Incidentally this is another proof of the structure theorem for orbits.

In fact, this proof is more general than our earlier one in that G is not assumed to be a connected group. We formulate this as a theorem.

THEOREM 12. *If G is a compact group acting on a k-dimensional orbit M, then M is locally the topological product of a k-cell by a compact zero dimensional set.*

13. Let G be a group defined by the sequence (A). Let G_i be the subgroup consisting of elements of the form

$$(e_1, e_2, \cdots, e_i, h_{i+1}, \cdots)$$

where e_i is the identity of H_i. This is a closed invariant subgroup and $G/G_i = H_i$.

LEMMA 9. *Let G be a group acting on an orbit G/G^* which is locally connected; that is, a finite set of manifolds. Then for some i, G_i is in G^*.*

Let G/G^* be determined by the sequence (B). For $i \geqq N$, f_i^* is a homeomorphism. For such a value of i let h_{i+1} be an element in $f_i^{-1}(e_i)$. The element h_{i+1} must be in H_{i+1}^* because $f_i(h_{i+1}) = f_i(e_{i+1})$ and if h_{i+1} were not in H_{i+1}^* then h_{i+1} and e_{i+1} would determine two different cosets of H_{i+1}^* going by f_i^* into the same coset of H_i^*. This would mean that f_i^* was not a homeomorphism.

COROLLARY 6'. *If G acts effectively on a locally connected orbit then G is a Lie group.*

Corollary 9. *If G acts on a space E the set of points on locally connected orbits form an F_σ.*

14. If G acts on a locally euclidean space E with all locally connected orbits there is defined at each point x of E a function $p(x)$ where $p(x)$ is the smallest integer p such that G_p (as in §13) is in G_x. Let L be the set of points in the space where $p(x)$ has an infinite least upper bound. The function is lower semi-continuous and integral valued so that L is a closed nowhere dense set.

Lemma 10. *If G acts effectively on a connected locally euclidean space E with all locally connected orbits and if L is void then $p(x)$ is bounded on E.*

It is clear, since L is void, that $p(x)$ is bounded at each point of E but the Lemma states that a uniform bound for all of E can be found.

Let O be a maximal open connected subset of E on which $p(x)$ is at most equal to k. Such sets exist for a properly chosen k by the nature of $p(x)$. Assume that O does not coincide with E. Then there is a point b on the boundary of O and not in O. Let U be a connected neighborhood of b on which $p(x)$ is bounded say by $r \geqq k$. The space $G(O + U)$ is connected and locally euclidean. Since G_r leaves every point of this space fixed, G/G_r acts on the space with the original orbits. Under the homomorphism $G \rightarrow G/G_r$, G_k goes into a Lie subgroup of G/G_r. This subgroup leaves all of O fixed and consequently it leaves all of $G(O + U)$ fixed. But then $p(x)$ is bounded at b by k. Hence O is not maximal. This shows that O and E coincide thus proving the Lemma.

15. Theorem 13. *If G acts effectively on a connected locally Euclidean space E and if all the orbits of G are locally connected, that is if each orbit is a finite set of manifolds, then G is a Lie group.*[7]

Assume that $p(x)$ is not bounded on E. Then by the preceding Lemma, L is not void. Let b be any point of L where $p(x \mid L)$, the function $p(x)$ considered only on L, is continuous. There is an open set O including b such that $p(x \mid L)$ is constant and equal to an integer k everywhere in $O \cap L$. Let R be any component of $E - L$ having points in O. There must be an integer $r \geqq k$ bounding $p(x)$ in R. Let $R_1 = G_k(R)$. The function $p(x)$ is bounded on R_1 and consequently no point of L can be an inner point of $\bar{R}_1$. Let c be a point of $\bar{R}_1 \cap L \cap O$. This point then is not an inner point of $\bar{R}_1$. Let V be a connected open set which contains c. As we have seen $V - R_1$ is not vacuous; it is separated from R_1 by L. For this reason the new transformation group of $R_1 + V$ about to be defined satisfies all the conditions for a transformation group.

If x is in R_1 and g is any element of G_k let $g[x] = g(x)$; if x is in $V - R_1$ let $g[x] = x$. The group G_r with G_k as thus defined leaves every point of $R_1 + V$

[7] The method of proof used here resembles that used by one of the authors in another connection. See American Journal of Mathematics, vol. 59, pp. 118–120.

fixed and therefore the Lie group G_k/G_r may be defined as a transformation group of $R_1 + V$ with the same orbits as for the group just defined. The group G_k/G_r leaves $V - R_1$ fixed; since this is an open subset of the connected locally euclidean space $R_1 + V$ we conclude that G_k/G_r leaves all of $R_1 + V$ fixed. Hence G_k as originally defined leaves all of R_1 fixed. The component R was an arbitrary one touching O and therefore every point x of $O - L$ is such that $p(x) \leqq k$. This shows that $p(x)$ is bounded at points of L in O and this contradicts the definition of L^*.

Another way of expressing this theorem is to say that the group G is effectively a Lie group, whenever it operates on a connected locally euclidean space with locally connected orbits. The proof could be attained more rapidly by assuming E to be compact and G connected.

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TRIANGULATED MANIFOLDS WHICH ARE NOT BROUWER MANIFOLDS[1]

BY STEWART S. CAIRNS

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1. The topological equivalence problems.

The several theorems established below touch upon the general question of the relationships among various classes of manifolds. We commence with five basic definitions.

(1) *Topological m-manifold* ($m = 0, 1, 2, \cdots$): A connected topological space which can be covered by a denumerable set of neighborhoods, each of which is an open m-cell.

(2) *Triangulable manifold*: A topological manifold which can be subdivided into the cells of a complex.

(3) *Star m-manifold*: A triangulated m-manifold on which the region covered by the star of any vertex is an m-cell.

(4) *Brouwer m-manifold*:[2] A star m-manifold for which the star of each vertex can be mapped homeomorphically into a euclidean m-space, E^m, so that the image of each m-cell of the star is an m-simplex.

(5) *Differentiable* (*analytic*) *m-manifold*: A topological m-manifold which can be covered with local coordinate systems having m-cells for domains, such that the transformation between any two of the systems is differentiable (analytic) with a non-vanishing jacobian.

Our considerations, appropriately modified, apply also to manifolds with boundaries and with partial boundaries, but we confine ourselves, for simplicity of statement, to the manifolds just defined.

One of the problems which the writer has studied is that of determining the extent to which any two of the above types of manifold are topologically equivalent in the sense that an arbitrary manifold of either type has a homeomorphic image of the other type. In so far as topological and triangulable manifolds are concerned, this problem is the general triangulation problem for manifolds. The topological equivalence question in all its generality is thus seen to be an extremely difficult one. There exist, however, a number of theorems which are worth noting. It is easy to verify, with the aid of known results, that all the above types of manifold are equivalent for $m < 3$. Whitney has shown[3]

[1] Presented to the American Mathematical Society, April 26, 1940. The results are also contained in a note of the same title in the Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 359-361.

[2] L. E. J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Mathematische Annalen 71 (1912), pp. 97-115.

[3] Hassler Whitney, *Differentiable manifolds*, these Annals, vol. 37 (1936), pp. 645-680.

that differentiable and analytic manifolds are topologically equivalent. The writer has shown[4] that the types of manifold numbered (2) to (5) above are topologically equivalent when $m = 3$, so that any 3-manifold which can be triangulated can be made analytic.

Each of the five classes of manifolds listed above is topologically equivalent to a subset of each preceding class. This statement is obvious, save for the part which asserts that every differentiable manifold is homeomorphic to a Brouwer manifold. This part of the statement follows readily from Whitney's imbedding theorem,[3] which shows that every differentiable manifold can be differentiably imbedded in a euclidean space, together with the writer's polyhedral approximation theorem,[5] which gives a Brouwer triangulation for manifolds so imbedded.

2. The theorems.

The results here developed are so closely related that they might be regarded as different aspects of one theorem, rather than as several distinct theorems. We list them separately, since the geometric content of each appears interesting. All of them will be deduced later (§4) from a certain basic lemma.

THEOREM I. *There exist star m-manifolds for every $m > 3$ which are not Brouwer manifolds.*

It is not known whether there exist *essentially* non-Brouwer manifolds; that is, manifolds which cannot be made into Brouwer manifolds by a suitable triangulation. This question is particularly significant, for an affirmative answer would imply (see H) the existence of manifolds which cannot be made differentiable.

THEOREM II. *Brouwer's definition[2] of an m-manifold is not invariant ($m > 3$) under subdivision.*

By a *polyhedral representation*, P^m, of a complex, K, we mean the image of the complex in a euclidean space under a homeomorphism such that each cell of K corresponds to a euclidean simplex on P^m.

THEOREM III. *Every triangulated m-sphere, $m < 3$, has a convex polyhedral representation in E^{m+1}. For every $m \geqq 3$, there exist triangulated m-spheres which have no convex, or even star-shaped, polyhedral representations in E^{m+1}.*

THEOREM IV. *The smallest n such that a given complex K has a polyhedral representation in E^n is not, for every K, invariant under subdivisions. In some cases, this n can be made alternately to increase and decrease during a sequence of successive subdivisions.*

THEOREM V. *For every $m > 3$, there exist stars of simplexes which, regarded as point sets, are m-cells, but which do not admit transversal $(n - m)$-planes no matter how they are polyhedrally represented in any E^n. Hence there exist star*

[4] S. S. Cairns, *Homeomorphisms between topological manifolds and analytic manifolds*, these Annals, immediately preceding the present article. We refer to the foregoing paper as H.

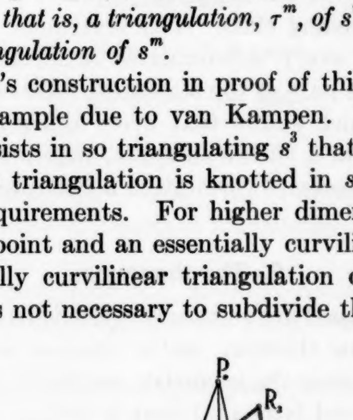
[5] S. S. Cairns, *Polyhedral approximations to regular loci*, these Annals, vol. 37 (1936), pp. 409–415.

manifolds which do not admit polyhedral representations in normal position (*see H*) *in any* E^n.

3. The basic lemma.

BASIC LEMMA. *For any* $m \geqq 3$, *there exists an essentially curvilinear triangulation of an m-simplex*, s^m; *that is, a triangulation*, τ^m, *of* s^m, *which is not isomorphic with any rectilinear triangulation of* s^m.

Aside from the writer's construction in proof of this lemma, there exists an easily comprehended example due to van Kampen. Van Kampen's method, in the case $m = 3$, consists in so triangulating s^3 that a 1-circuit composed of just three 1-cells of the triangulation is knotted in s^3. Such a triangulation obviously fulfills the requirements. For higher dimensions, it is sufficient to note that the join of a point and an essentially curvilinear triangulation of an s^{m-1} affords an essentially curvilinear triangulation of an s^m. We note, for later purposes, that it is not necessary to subdivide the boundary, β^{m-1}, of s^m



in constructing an essentially curvilinear triangulation, for an arbitrarily triangulated simplex, s_0^m, can be put inside s^m, and the triangulation then extended over the rest of s^m without subdividing β^{m-1} or further subdividing s_0^m.

The writer's example is illustrated by the accompanying figure. Only two vertices, P_0 and P_1, of s^3 are shown. Interior to s^3 are the four points Q_i ($i = 1, 2, 3, 4$), joined by 1-simplexes as indicated in the diagram. There are three 2-simplexes incident with P_0 and having Q_1Q_2, Q_2Q_3, and Q_3Q_1 as respective opposite edges. There are four 2-simplexes incident with P_1 and having Q_1Q_2, Q_2Q_4, Q_4Q_3, and Q_3Q_1 as respective opposite edges. This makes seven 2-simplexes, which are then all joined by 3-simplexes to a point Q_0. The seven resulting 3-simplexes, with bounding cells, will be a subcomplex, U, of the required triangulation. Suppose the points Q_i so located that U can be made rectilinear. Once the seven 2-simplexes opposite Q_0 are fixed, the domain, $D(Q_0)$, of possible positions for Q_0 is the intersection of eight half-spaces bounded, respectively, by the planes of the seven 2-simplexes and the plane of the points (Q_2, Q_3, Q_4).

The next step is the introduction of a 1-circuit, $\gamma = R_1R_2R_3$, composed of three 1-simplexes, where γ cannot be shrunk to a point in $s^3 - U$. We then ensure an essentially curvilinear triangulation, τ^3, by requiring that τ^3 contain as a subcomplex the cells U, the cells of γ, and three 1-cells joining Q_0 to R_1, R_2, and R_3, respectively. For, with U rectilinear, the points which can be joined to Q_0 by line-segments avoiding U fill out the intersection, ρ, of s^3 with the trihedral angle $(Q_0 ; Q_2Q_3Q_4)$. Note that (Q_2, Q_3, Q_4) are always outside or on the boundary of the convex region $D(Q_0)$ mentioned above, to which Q_0 is confined. The region ρ is convex. Hence if it contained R_1, R_2, and R_3 it would have to contain the entire 2-simplex which they determine. But this 2-simplex contains inner points of U, which obviously cannot be joined to Q_0 by line-segments avoiding U. Thus a contradiction is obtained. To complete the argument, it remains only to verify that the remainder of the triangulation τ^3 can be constructed without subdividing any of the cells thus far introduced. The writer has done this but omits the details since they involve no difficulties. In the treatment for the general $m > 3$, the essential change is the replacement of Q_2Q_3 by an $(m - 2)$-simplex.

4. Establishment of the theorems

Commencing with Theorem III, we note that the cases $m < 3$ are easy to verify. In the case $m \geqq 3$, a triangulated m-sphere, τ^m, of the sort required is obtained from the boundary, β^m, of an $(m + 1)$-simplex by subjecting each m-simplex s_i^m $(i = 1, \cdots, m + 1)$ of β^m to an essentially curvilinear triangulation, τ_i^m, where τ_i^m does not subdivide the boundary of s_i^m. Suppose τ^m polyhedrally imbedded in an E^{m+1} so as to be star-shaped relative to a point O. If, from O, we project τ^m into a geodesic triangulation of an m-sphere about O, then the image of at least one of the subcomplexes τ_i^m will be a proper subset of a hemisphere. Hence this subcomplex could be homeomorphically projected from O, contrary to its definition, into a rectilinear triangulation of an m-simplex on some m-plane. Thus Theorem III is established.

To obtain Theorem I, given any triangulated m-manifold $(m > 3)$, we can subdivide it so that the boundary of the star of some vertex shall admit no star-shaped polyhedral representation in any E^m. The subdivided manifold is then not a Brouwer manifold.

Theorem II follows at once. Note that the property of being a Brouwer manifold can be alternately lost and regained under successive subdivisions.

In the case of Theorem IV, the simplest illustration is where K is a 3-simplex plus bounding simplexes. Then the smallest n is three. By an essentially curvilinear triangulation, we can raise n to four. A further subdivision can then be given reducing the smallest n to three, and so forth.

Theorem V is an immediate consequence of Theorem I together with the theorem and lemma in §3 of H.

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HOMEOMORPHISMS BETWEEN TOPOLOGICAL MANIFOLDS AND ANALYTIC MANIFOLDS[1]

By Stewart S. Cairns

(Received September 25, 1939)

1. Existence of the homeomorphisms. By a *topological m-manifold*, M, $(m = 0, 1, \cdots)$ we mean a connected topological space which can be covered with a denumerable set of neighborhoods, each of which is an m-cell. We will employ, on M, various coordinate systems, each having an m-cell for domain and each defined by a homeomorphism between its domain and a region in a euclidean m-space, E^m.

Consider a set, X, of coordinate systems whose domains cover M. We will say that M is *analytic* (*in terms of the systems* X) if every transformation

$$v_i = v_i(x) \qquad (i = 1, \cdots, m) \tag{1.1}$$

between two of the systems, $(x) = (x_1, \cdots, x_m)$ and (v), whose domains overlap, is analytic with a non-vanishing jacobian.

Let $(y) = (y_1, \cdots, y_n)$ denote a coordinate system in E^n. A *topological m-manifold in* E^n will mean a set of points

$$M\colon y_i = y_i(p) \qquad (i = 1, \cdots, n) \tag{1.2}$$

where (1) p is a variable point on a topological m-manifold, M', and (2) the correspondence (1.2) between M and M' is a homeomorphism. Let X be a set of coordinate systems whose domains cover M'. As p ranges over the domain of any system (x), the functions $y_i(p)$ can be interpreted as functions of (x). If all such functions are analytic and if every functional matrix $(\partial y_i/\partial x_j)$ is of rank m on its domain, then M will be called an *analytic manifold in* E^n (*in terms of the systems* X *and the correspondence* (1.2)).

Consider a point set, S, in E^n. A k-plane, π^k $(k \geqq 1)$, through a point p of S will be called *transversal to* S *at* p if it makes angles bounded away from zero with the secant lines of some neighborhood of p on S. Any plane, π^k, is called *transversal to* S (*in the large*) if it makes angles bounded away from zero with all the secant lines of S.

We will say that a topological m-manifold, M, in E^n is in *normal position* if it is possible to define, through each point p of M, an $(n - m)$-plane, $\pi^{n-m}(p)$, in such a way that (1) $\pi^{n-m}(p)$ varies continuously with p and (2) $\pi^{n-m}(p)$ is transversal to M at p.

Suppose the topological m-manifold, M, can be subdivided into the cells of a

[1] Presented to the American Mathematical Society; March 27, 1937, February 25, 1939, and October 28, 1939.

simplicial complex. It can then be mapped by a homeomorphism into a polyhedral complex, P^m, in an E^n $(n > 2m)$, where the faces of P^m are euclidean simplexes corresponding to the cells into which M is subdivided.

THEOREM I. *Given a topological m-manifold, M, there exists a set of coordinate systems in terms of which M is analytic with an analytic Riemannian metric, if and only if M can be triangulated so as to have a polyhedral representation, P^m, in normal position in some E^n.*

This theorem, in so far as the sufficiency of the condition is concerned, is a consequence of the following.

THEOREM[2] II. *Arbitrarily near any normal position of P^m, there exists an analytic manifold in E^n, homeomorphic to P^m.*

Part of this paper is devoted to an investigation of conditions under which a polyhedral manifold P^m can be put into normal position. By showing this to be always possible when $m = 3$, we obtain the following result. The cases $m < 3$ can easily be dealt with by known methods.

THEOREM III. *If a topological 3-manifold, M, can be triangulated, then there exists a set of coordinate systems in terms of which M is analytic and has an analytic Riemannian metric.*

2. Normal positions and general positions. We now establish the necessity of the conditions in Theorem I. Any analytic m-manifold, M, has a homeomorph, M', which is analytic in some euclidean space E^n (DM, Theorem I). The writer has shown[3] that M' can be so triangulated into cells (σ) that (1) the vertices of each m-cell determine a non-degenerate m-simplex and (2) the totality of the simplexes so determined is a polyhedral manifold, P^m, homeomorphic to M' in such a way that corresponding m-cells have identical vertices and that the tangent m-plane to M' at any point of a cell, σ^m, of (σ) differs arbitrarily little in direction from the m-plane of the corresponding face of P^m. Now suppose (p, q) are corresponding points on (P^m, M') respectively. If $\pi^{n-m}(p)$ is the $(n - m)$-plane through p parallel to the $(n - m)$-plane normal to M' at q, and if P^m is a sufficiently close approximation to M', then $\pi^{n-m}(p)$ is transversal to P^m as required by the definition of normal position. For, since the faces of P^m are approximately tangent to M', the directions of the secant lines of any neighborhood on P^m are approximately the same as in the case of the corresponding neighborhood on M'.

The sufficiency proof for Theorem I will not be complete until §8.

Consider an arbitrary triangulated topological m-manifold, M. We will assume first[4] that there exists an upper bound to the number of cells in a star

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[2] Our proof of Theorem II will involve methods due to Hassler Whitney. See *Differentiable manifolds*, Annals of Mathematics, vol. 37 (1936), pp. 645-680. This paper will be referred to as DM.

[3] *Polyhedral approximations to regular loci*, Annals of Mathematics, vol. 37 (1936), pp. 409-415.

[4] In §9 a method is given which does not involve this hypothesis.

on M. The assumption enables us to imbed P^m in an E^n, for n sufficiently large, so that the vertices on each star of simplexes are linearly independent. P^m is then said to be in *general position.*

LEMMA. *If it is possible to put P^m into normal position, then there exists a general position which is also a normal position.*

PROOF. We commence with an auxiliary result.

(A) If a manifold, M, is in normal position in E^ν, then it is in normal position in any E^n which contains E^ν as a subspace.

For, suppose that $\pi^{\nu-m}(p)$ in E^ν is transversal to M at p and that $\pi^{n-\nu}(p)$ in E^n is transversal to E^ν at p. Then it follows from our definitions that $\pi^{n-m}(p)$, determined by $\pi^{\nu-m}(p)$ and $\pi^{n-\nu}(p)$ is transversal to M at p. It remains only to require that $\pi^{n-\nu}(p)$ be continuous in p on M. We might, for example, use the $(n - \nu)$-plane normal to E^ν at p.

Now let P^m be in normal position in $E^\nu \subset E^n$, n being so large that E^n can contain P^m in general position. Then P^m can be brought into a general position, $*P^m$, by arbitrarily small displacements of its vertices. Let barycentric coordinates be introduced on the simplexes of P^m and, in precisely the same way, on the simplexes of $*P^m$. Two points on $(P^m, *P^m)$, respectively, will correspond if their coordinates are the same. Suppose $\pi^{n-m}(p)$ is transversal to a certain neighborhood, $N(p)$, on P^m. The directions of the secant lines of the corresponding neighborhood, $N(p^*)$, on $*P^m$ can be made arbitrarily close to those of $N(p)$ by suitable restrictions on the displacements carrying P^m into $*P^m$. Hence it can be arranged that $\pi^{n-m}(p^*) \parallel \pi^{n-m}(p)$ shall be transversal to $*P^m$ at p^*, as required by the definition of normal position.

3. Planes transversal to Brouwer stars. The triangulated manifold M, or its representation P^m, will be called a *Brouwer manifold*[5] if the star of each vertex on P^m can be mapped into an E^m by a piecewise linear homeomorphism; that is, a homeomorphism which is linear on each simplex of the star.

THEOREM IV. *No P^m can be put into normal position unless it is a Brouwer manifold.*

This follows immediately from the first sentence in the lemma below.

For $m > 3$, it is unknown[6] whether every triangulated m-manifold is a Brouwer manifold. We show, in §8, that this is surely true for $m = 3$. It is obvious for $m < 3$.

Let (π^m, π^{n-m}) be planes, of the indicated dimensions, transversal to each other in the euclidean space E^n. Consider the plane parallel to π^{n-m} through any point p in E^n. This plane meets π^m in a point, p', which will be called the *π^{n-m}-projection of p on π^m.* The locus of p' as p ranges over a point set S will

[5] Brouwer, *Über Abbildungen von Mannigfaltigkeiten,* Mathematische Annalen 71 (1912), pp. 97–115.

[6] Since this was written, examples of non-Brouwer triangulated manifolds have been constructed. See *Triangulated manifolds which are not Brouwer manifolds,* immediately following the present article.

be referred to as the π^{n-m}*-projection of* S *on* π^m. This same expression will be used for the mapping of S onto π^m in which p and p' correspond. The following result is then obvious.

(A) *The* π^{n-m}*-projection of* S *onto* π^m *is a homeomorphism if* π^{n-m} *is transversal to* S. *As a partial converse, if* S *is bounded and the* π^{n-m}*-projection of the closure,* $\bar{S}$, *of* S *is a homeomorphism, then* π^{n-m} *is transversal to* $\bar{S}$ *and hence to* S.

Let $S^k = S^k(s^j)$ be a set of simplexes of dimensions $(j, \cdots, k)$ in E^n, where (1) S^k is the star of a j-simplex, s^j, and (2) S^k, regarded as a point set, is a k-cell. We will refer to S^k as a *Brouwer k-star* if it has a piecewise linear homeomorph in an E^k.

LEMMA. *If* $S^k(s^j)$ *can be put into normal position, it is a Brouwer star. Furthermore, every general position of a Brouwer star is also a normal position.*

As a first step in the proof, we make the following easily verified statement.

(B) *If* N *is an arbitrary neighborhood on* $S^k(s^j)$ *of a point on* s^j, *then every secant of* $S^k(s^j)$ *is parallel to a secant of* N.

It follows at once that, when S^k is in normal position, any $(n - k)$-plane π^{n-k} transversal to S^k at a point of s^j is also transversal to S^k in the large. The π^{n-k}-projection of S^k onto any π^k transversal to π^{n-k} affords a piecewise linear homeomorphism as required by the definition of Brouwer star. This establishes the first statement in the lemma.

Consider, now, a Brouwer star, S^k, in general position in E^n. By definition, there exists a piecewise linear homeomorphism, $\Lambda = \Lambda(S^k)$, mapping S^k onto an E^k. Let $(P_0, \cdots, P_\nu)$ denote the vertices of S^k, the notation being so assigned that $(P_0, \cdots, P_k)$ are the vertices of some k-cell of S^k. Let $(y) = (y_1, \cdots, y_n)$ be a rectilinear coordinate system in E^n, relative to which P_0 is the origin and P_i $(i = 1, \cdots, \nu)$ is unit point on the y_i-axis.

We restrict Λ so that it will map S^k onto the coordinate $(y_1, \cdots, y_k)$-plane with P_i self-corresponding $(i = 0, \cdots, k)$. Let $Q_j : (a_{j1}, \cdots, a_{jk}, 0, \cdots, 0)$ denote the image of P_j $(j = k + 1, \cdots, \nu)$ under Λ, and consider the transformation of coordinates

$$
\begin{aligned}
x_i &= y_i + \sum_{j=k+1}^{\nu} a_{ji} y_j && (i = 1, \cdots, k) \\
x_j &= y_j && (j = k + 1, \cdots, n).
\end{aligned}
\tag{3.1}
$$

In terms of the coordinate system (x), P_0 is still the origin, P_i $(i = 1, \cdots, k)$ is the unit point on the x_i-axis, and the following are the coordinates of the remaining P's and of the Q's:

$$
\begin{aligned}
&P_j:(a_{j1}, \cdots, a_{jk}, 0, \cdots, 0, x_j = 1, 0, \cdots, 0) \\
&Q_j:(a_{j1}, \cdots, a_{jk}, 0, \cdots, 0)
\end{aligned}
\qquad (j = k + 1, \cdots, \nu).
\tag{3.2}
$$

Hence, if π^{n-k} denote the $(x_{k+1}, \cdots, x_n)$-plane, then Λ is the π^{n-k}-projection onto the plane of $(P_0, \cdots, P_k)$. Therefore, by result (A), π^{n-k} is transversal to S^k, and the proof is complete.

4. Spaces of transversal planes. Let S^k be a Brouwer k-star in general position in E^ν. We will denote with $\Pi(S^k, E^\nu)$ the topological space each of whose points is a system of parallel $(\nu - k)$-planes transversal to S^k in E^ν, continuity being defined in terms of direction cosines.[7] The following statements are easy to verify.

(*A*) *If*

$$S^k \subset E^\nu \subset E^n \tag{4.1}$$

then $\Pi(S^k, E^n)$ *is the set of* $(n - k)$*-planes in* E^n *which intersect* E^ν *in planes of the set* $\Pi(S^k, E^\nu)$. [Compare the proof of §2(A)].

(*B*) *If* S^k_0 *is a subset of* S^k, *then*

$$\Pi(S^k, E^n) \subset \Pi(S^k_0, E^n). \tag{4.2}$$

We will refer to two topological spaces, Σ_1 and Σ_2, as *β-equivalent*[8] provided the following statement holds: For each j ($j = 0, 1, \cdots$) every (singular or non-singular) j-sphere in Σ_1 bounds a $(j + 1)$-cell if and only if every j-sphere in Σ_2 also bounds a $(j + 1)$-cell.

Lemma. *Any space* $\Pi(S^m(s^k), E^n)$ *is β-equivalent to a certain space* $\Pi(S^{m-k}(s^0), E^{n-k})$.

Proof. We take s^0 as the barycenter of s^k, and E^{n-k} as the $(n - k)$-plane normal to s^k at s^0. The star $S^{m-k} = S^{m-k}(s^0)$ is defined as the projection of $S^m = S^m(s^k)$ onto E^{n-k}. We commence with the following auxiliary result.

(*C*) *Suppose* π^{n-m} *belongs to* $\Pi(S^m, E^n)$, *and let* π^k *denote the k-plane of* s^k. *Then the plane* π^{n-m+k} *determined by* π^{n-m} *and* π^k *belongs to* $\Pi(S^{m-k}, E^n)$.

In the first place, we note that the secants of S^{m-k} are a subset of those of S^m. With the aid of §3(*B*), one can verify that if l is a line in E^n whose π^k-projection is a secant of S^{m-k}, then l is parallel to a secant of S^m. Since every line on π^{n-m+k} is parallel to π^k or else has the same π^k-projection as a line on π^{n-m}, it follows that the only secants of S^m parallel to π^{n-m+k} are also parallel to π^k. Hence no secant of S^{m-k} is parallel to π^{n-m+k}. Since the secants of S^{m-k} are a closed set, result (*C*) follows at once. It also follows that $\Pi(S^{m-k}, E^{n-k})$ consists of the intersections of E^{n-k} with planes such as π^{n-m+k}.

An $(n - m)$-plane π^{n-m} on π^{n-m+k} belongs to $\Pi(S^m, E^n)$ if it contains no line parallel to π^k. Any subset of $\Pi(S^m, E^n)$ consisting of planes whose angles with π^k all exceed $\vartheta > 0$ can therefore be homotopically deformed in $\Pi(S^m, E^n)$ so that each π^{n-m} remains in a single plane such as π^{n-m+k} and is carried into its π^k-projection on E^{n-m}. Now suppose every sphere in $\Pi(S^m, E^n)$ bounds a cell. Since $\Pi(S^{m-k}, E^{n-k})$ is a subset of $\Pi(S^m, E^n)$, any sphere, β^j, in the former space bounds a cell, σ^{j+1}, in the latter. Some deformation of the sort just

[7] S. S. Cairns, *The direction cosines of a p-space in euclidean n-space*, American Mathematical Monthly, vol. 39 (1932), pp. 518–523.

[8] The stronger condition of complete homology equivalence (cf. Alexandroff-Hopf, *Topologie I*, 1935) might be established for the spaces we treat. However, we need only β-equivalence.

described will leave β^j fixed and will carry σ^{j+1} into a $(j + 1)$-cell in $\Pi(S^{m-k}, E^{n-k})$. Hence every sphere in $\Pi(S^{m-k}, E^{n-k})$ bounds a cell in $\Pi(S^{m-k}, E^{n-k})$.

Suppose conversely that, in $\Pi(S^{m-k}, E^{n-k})$, every sphere bounds a cell: in other words, can be shrunk to a point. An arbitrary sphere in $\Pi(S^m, E^n)$ can be deformed, as above, into a sphere in $\Pi(S^{m-k}, E^{n-k})$ and then further shrunk to a point. This completes the establishment of the lemma.

5. β-equivalent spaces of maps. Corresponding to a Brouwer k-star, S^k, we define a space of mappings $\Lambda(S^k)$ as follows. Each point of $\Lambda(S^k)$ can be represented by a piecewise linear homeomorphism of S^k into an E^k. Two such homeomorphisms represent the same point of $\Lambda(S^k)$ if and only if one can be carried into the other by a linear transformation of E^k. Let $(P_1, \cdots, P_\nu)$ be the vertices of S^k and $(P'_1, \cdots, P'_\nu)$ their respective images under a piecewise linear homeomorphism representing a point, λ_0, of $\Lambda(S^k)$. A neighborhood, $N(\lambda_0)$, in $\Lambda(S^k)$ will correspond as follows to any set of neighborhoods, $N_i(P'_i)$ $(i = 1, \cdots, \nu)$, in E^k: A point of $\Lambda(S^k)$ belongs to $N(\lambda_0)$ if and only if it can be represented by a piecewise linear homeomorphism carrying P_i into $N_i(P'_i)$ $(i = 1, \cdots, \nu)$. Thus $\Lambda(S^k)$ is defined as a topological space.

LEMMA 5.1. *There exists a homeomorphism between $\Lambda(S^k)$ and $\Pi(S^k, E^\nu)$ provided* (1) *S^k is in general position in E^ν and* (2) *no n-plane with $n < \nu$ contains S^k.*

PROOF. Let E^k be determined by the vertices $(P_0, \cdots, P_k)$ of a particular k-simplex of S^k. Since two piecewise linear homeomorphisms of S^k into E^k represent the same point of $\Lambda(S^k)$ if they are related by a linear transformation of E^k, we can obtain unique representations for the points of $\Lambda(S^k)$ by stipulating that $(P_0, \cdots, P_k)$ be self-corresponding. Comparing the proof of §3, Lemma, we see that a homeomorphism between $\Pi(S^k, E^\nu)$ and $\Lambda(S^k)$ is defined if each element $\pi^{\nu-k}$ of the former space be associated with the $\pi^{\nu-k}$-projection of S^k onto E^k.

LEMMA 5.2. *If $E^n \supset E^\nu$, then $\Pi(S^k, E^n)$ is β-equivalent to $\Pi(S^k, E^\nu)$ and hence to $\Lambda(S^k)$.*

This can be proved, on the basis of Lemma 5.1, by reasoning as in the proof of §4, Lemma.

Given $S^k = S^k(s^j)$, let p_0 be the barycenter of s^j. We will denote with $\Lambda^0(S^k)$ the subspace of $\Lambda(S^k)$ consisting of those elements which map all the boundary vertices of S^k onto the unit $(k - 1)$-sphere in E^k about the image of p_0.

LEMMA 5.3. *The spaces $\Lambda(S^k)$ and $\Lambda^0(S^k)$ are β-equivalent.*

PROOF. We represent all the points of $\Lambda(S^k)$ by elements mapping p_0 into the origin, O, in E^k and mapping the vertices of some particular k-cell into prescribed images on the unit $(k - 1)$-sphere S^{k-1} about O. Let λ_0 denote any element of $\Lambda(S^k)$, thus restricted, and let q_0 be the image of any vertex, p, of S^k under λ_0. Let q_1 denote the intersection of S^{k-1} with the ray Oq_0. We then denote with λ_t that element of $\Lambda(S^k)$ which carries each vertex p into the point q_t on the segment q_0q_1 such that $q_0q_t = t \cdot q_0q_1$. As t increases from 0 to 1, λ_t

defines a deformation of λ_0 into λ_1. By applying this deformation simultaneously to all the elements λ, we deform the whole space $\Lambda(S^k)$ into $\Lambda^0(S^k)$. Using this deformation, one can complete the proof as in the case of §4, Lemma.

6. The space of triangulations $T(\tau^{k-1})$. Given a Brouwer star $S^k = S^k(s^0)$, let the points of $\Lambda^0(S^k)$ be represented by homeomorphisms carrying the vertices $(s^0, P_1, \cdots, P_k)$ of some k-cell into specified images $(O, Q_1, \cdots, Q_k)$ in E^k. Let Σ^k denote the image of S^k under some such homeomorphism, λ. By definition of $\Lambda^0(S^k)$, the boundary, B^{k-1}, of Σ^k has all its vertices on the unit sphere, S^{k-1}, about O. The central projection from O onto S^{k-1} maps B^{k-1} into a geodesic triangulation, τ^{k-1}, of S^{k-1}; that is, one in which each cell appears as a simplex relative to some local coordinate system in which arcs of great circles are represented as straight lines. This implies that the closure of each cell of τ^{k-1} is on an open hemisphere of S^{k-1}, in other words that each 1-cell is less than 180°. Let $T(\tau^{k-1})$ be the following topological space. Its points are geodesic triangulations of S^{k-1} homeomorphic to τ^{k-1} with Q_i $(i = 1, \cdots, k)$ self-corresponding. If τ_0^{k-1} is any such triangulation and Q_i $(i = 1, \cdots, \nu)$ are its vertices, then a neighborhood, $N(\tau_0^{k-1})$, in $T(\tau^{k-1})$ corresponds as follows to a given set of neighborhoods $N_j(Q_j)$ $(j = k + 1, \cdots, \nu)$ of the points Q_j on S^{k-1}: $N(\tau_0^{k-1})$ consists of all elements of $T(\tau^{k-1})$ for which the vertex corresponding to Q_j lies in $N_j(Q_j)$.

The following lemma is a direct consequence of our definitions.

LEMMA. *The spaces $T(\tau^{k-1})$ and $\Lambda^0(S^k)$ are homeomorphic under the correspondence induced by the central projection from O.*

7. A sufficient condition in the normal position problem. THEOREM V. *A sufficient condition that it be possible to put a Brouwer m-manifold into normal position is that, in every space $\Pi(S^k(s^0), E^\nu)$ [or $\Lambda(S^k)$ or $T(\tau^{k-1})$], every $(m - k - 1)$-sphere bound an $(m - k)$-cell*[9] *$(k = 0, \cdots, m - 1)$.*

The proof will occupy this section and the next.

(A) Let η be a positive constant less than $1/(m + 1)$. If s^j be any j-simplex, $j \leqq m$, then the *η-core*, γ^j, of s^j will mean the set of points where all the barycentric coordinates for s^j exceed η. In the case $j = 0$, we have $\gamma^0 = s^0$.

We now consider an m-simplex, s^m, and define on it certain neighborhoods, $N(\gamma^j)$, where γ^j is the η-core of a typical bounding simplex, s^j, of s^m. The definition will be recurrent in $j = 0, \cdots, m$. To define $N(\gamma^0)$, we choose barycentric coordinates $(u_0, \cdots, u_m)$ on s^m so that $u_i = 0$, $(i = 1, \cdots, m)$ at γ^0. We then make the definition

$$N(\gamma^0)\colon O \leqq u_i \leqq \eta \qquad (i = 1, \cdots, m). \tag{7.1}$$

To define $N(\gamma^j)$, $O < j < m$, choose the barycentric coordinates so that $u_i = 0$

---

[9] For $k = 0$, the space $T(\tau^{k-1})$ is vacuous, and the condition becomes trivial.

$(i = j + 1, \cdots, m)$ on γ^j. Assume the $N(\gamma^k)$ all defined $(k = 0, \cdots, j - 1)$. We define a region $N'(\gamma^j)$ as follows:

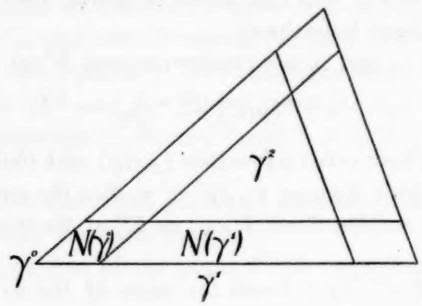
$$N'(\gamma^j)\colon\ 0 \leqq u_i \leqq \eta \qquad (i = j + 1, \cdots, m) \tag{7.2}$$

and then make the definition

$$N(\gamma^j) = \overline{N'(\gamma^j) - \sum_{k<j} N(\gamma^k)}, \tag{7.3}$$

the summation being over the η-cores of all bounding k-simplexes of s^m $(k < j)$. Then $N(\gamma^j)$ is a closed, box-like, m-dimensional region with γ^j for one face. The simplex s^m is covered by γ^m plus the regions $N(\gamma^j)$ $(j < m)$. These regions are distinct, save for common faces. The accompanying figure illustrates the definitions when $m = 2$.

Any core γ^j, $0 \leqq j < m$, is parallel to a certain bounding simplex, $*\gamma^j$, of γ^m. In case $j = 0$, $*\gamma^j$ will denote the vertex of γ^m nearest γ^0. Let σ^{m-j-1} denote the



bounding simplex of γ^m opposite $*\gamma^j$. Consider any point, q, on γ^j. We will denote with $B^{m-j}(q)$ the intersection of $N(\gamma^j)$ with the $(m - j)$-plane determined by q and σ^{m-j-1}. Then $B^{m-j}(q)$ is a box-like $(m - j)$-dimensional point set. As q ranges over the closure of γ^j, the sets $B^{m-j}(q)$ fill out the region $N(\gamma^j)$ in continuous one-to-one fashion.

(B) If q is on the boundary of γ^j, then $B^{m-j}(q)$ is common to the boundaries of $N(\gamma^j)$ and some $N(\gamma^k)$ where $k < j$.

Now let s^m be any m-simplex of P^m and s^j any bounding simplex of s^m, with γ^j denoting its η-core. We will then use the notation

$$\begin{aligned} \mathfrak{N}(\gamma^j) &= \sum_{s^m \epsilon S(s^j)} N(\gamma^j) \\ \mathfrak{B}(q) &= \sum_{s^m \epsilon S(s^j)} B(q) \qquad q \text{ on } \gamma^j \end{aligned} \tag{7.4}$$

where $S(s^j)$ is the star of s^j on P^m.

Our proof will consist in assuming the condition of the theorem and constructing suitable transversal planes $\pi^{n-m}(p)$. We take P^m in general position in E^n. The method will be recurrent, with the following basic hypothesis.

HYPOTHESIS I. *For some value j of the set $(1, \cdots, m)$, a plane $\pi^{n-m}(p)$ has been defined so as to vary continuously with p on the sum of the neighborhoods $\mathfrak{N}(\gamma^k)$ $(k < j)$ and so as to be transversal to P^m at p.*

The initial step of the recurrency falls into two parts. We first select arbitrary transversal planes, $\pi^{n-m}(p)$, at the vertices $(s_1^0, s_2^0, \cdots) = (\gamma_1^0, \gamma_2^0, \cdots)$ [see (A) above] of P^m. This is possible, by §3, Lemma, since P^m is a Brouwer manifold. We then define $\pi^{n-m}(p)$ on the $\mathfrak{N}(\gamma^0)$ by the requirement

$$\pi^{n-m}(p) \parallel \pi^{n-m}(\gamma_i^0), \qquad p \text{ on } \mathfrak{N}(\gamma_i^0), \tag{7.5}$$

together with the requirement that $\pi^{n-m}(p)$ pass through p. The verification of hypothesis I for $j = 1$ depends on §4 (B), to be applied where $k = m$, $S^k = S(s_i^0)$, and S_0^k is the star of any simplex of P^m incident with s_i^0.

8. The general step of the recurrency. Hypothesis I would be a sufficient basis for the proof of Theorem V. The following hypotheses are used to secure regularity restrictions which will enable us to prove Theorem II. The initial step in §7 satisfies all these hypotheses.

HYPOTHESIS II. *If p_1 and p_2 are common to any $\mathfrak{B}^k(q)$, then*

$$\pi^{n-m}(p_1) \parallel \pi^{n-m}(p_2). \tag{8.1}$$

HYPOTHESIS III. *There exists a function $\xi_{j-1}(p)$ such that, if $R(p)$ denote the set of points on $\pi^{n-m}(p)$ within distance $\xi_{j-1}(p)$ of p, then the sets $R(p)$ fill out, in one-to-one fashion, a closed neighborhood, R_{j-1}, in E^n of the inner points of the $\mathfrak{N}(\gamma^k)$ $(k < j)$.*

HYPOTHESIS IV. *If $\pi^{n-m}(p')$ denote the plane of the set $\pi^{n-m}(p)$ through any point p' of R_{j-1}, then $\pi^{n-m}(p')$, regarded as a mapping of R_{j-1} into the space of all $(n - m)$-planes in E^n, is differentiable.*[10]

The general step of the recurrency extends the definition of $\pi^{n-m}(p)$ over the neighborhoods $\mathfrak{N}(\gamma^j)$. We break this step into two parts. In the first part, we extend the definition over a typical η-core, γ^j. In the second part, omitted when[11] $j = m$, we extend it over the rest of $\mathfrak{N}(\gamma^j)$.

The definition of $\pi^{n-m}(p)$ maps $s^j - \gamma^j$ differentiably[10] into $\Pi(S^m(s^j), E^n)$, which is β-equivalent to some space $\Pi(S^{m-j}(s^0), E^{n-j})$ [see §4, Lemma]. Since the boundary of γ^j is a $(j - 1)$-sphere, the condition of Theorem V, read with $k = m - j$, implies that the mapping of $s^j - \gamma^j$ can be extended over γ^j. As a result of Theorem 7 in *DM*, the extension can be made so as to give a differentiable[10] mapping of the whole of s^j into $\Pi(S^m(s^j), E^n)$. By such a mapping, we extend the definition of $\pi^{n-m}(p)$ over γ^j.

If, now, q is any point of γ^j, and p is any point of $\mathfrak{B}^{m-j}(q)$, $\pi^{n-m}(p)$ will mean the $(n - m)$-plane through p parallel to $\pi^{n-m}(q)$. This completes the definition of $\pi^{n-m}(p)$ on the neighborhoods $\mathfrak{N}(\gamma^j)$.

---

[10] This term has meaning here, because $\Pi(S^m(s^j), E^n)$ is a subspace of the space of all m-planes in E^n, and this latter space can be interpreted as an analytic manifold in some euclidean space (cf DM, §24).

[11] The proof of the theorem is completed with the first part of the step $j = m$.

The preservation of Hypotheses II and IV is an immediate consequence of the construction. In Hypothesis I, the transversality requirement is easy to verify, for the value j, with the aid of §4 (B), and the continuity requirement follows with the aid of §7 (B) and Hypothesis II. In establishing the preservation of Hypothesis III, it is convenient to consider the two parts of our general step: (1) the extension of $\pi^{n-m}(p)$ over γ^j and (2) the extension over the rest of $\mathfrak{N}(\gamma^j)$. The preservation during part (1) can, since Hypothesis IV is preserved, be proved by the methods of DM, Lemma 21. During part (2), Hypothesis III is preserved by virtue of the parallelism requirement in Hypothesis II.

9. Completion of the proofs of Theorems I and II. (*A*) *If a* P^m *be in normal position, then transversal planes* $\pi^{n-m}(p)$ *can be constructed by the recurrency in* §§7 *and* 8.

For, if $\pi'(p)$ denote any set of transversal planes relative to which P^m is in normal position, then one can construct, using the methods of the recurrency with η sufficiently small, an arbitrarily close approximation, $\pi^{n-m}(p)$, to $\pi'(p)$.

We are now ready to establish Theorem II, thus incidentally completing the proof of Theorem I. We assume P^m in normal position, with transversal planes $\pi^{n-m}(p)$ defined as in §§7 and 8. Hypotheses III and IV impose conditions which permit the application of Parts IV and V in DM, read with the following substitutions.

(1) The differentiable manifold, M, in E^n is to be replaced by the polyhedral manifold, P^m, in normal position in E^n.

(2) $\pi^{n-m}(p)$ plays the role of the plane $P(p)$, approximately normal to M at a point p.

(3) For a given point p on P^m, let $s^j(p)$ denote the simplex containing p. Assuming a fixed numbering $(s_1^m, s_2^m, \cdots)$ for the m-simplexes of P^m, let s_i^m denote the m-simplex with the smallest subscript belonging to the star of $s^j(p)$. The m-plane, $\tau(p)$, of s_i^m replaces the tangent m-plane, T, to M at a point p.

The relevant parts of Whitney's work can be outlined as follows, in their application to the construction of an analytic manifold M^* homeomorphic to P^m. First, an analytic $(n-1)$-manifold S, surrounding P^m, is defined. This is done with the aid of a function, $\Phi'(p)$, continuous in $R(P^m) = R_m$ (see Hypothesis III above), zero on P^m, and positive and analytic in $R(P^m) - P^m$. From Φ', there is subtracted a small positive analytic function, $\omega(p)$, such that the equation

$$\Phi'(p) - \omega(p) = 0 \tag{9.1}$$

determines a suitably restricted[12] analytic manifold, S. This manifold is such that if π^{n-m} passes through a point p in $R(P^m)$ and has direction cosines sufficiently close to those of $\pi^{n-m}(p)$ [cf. Hypothesis IV above], then (1) π^{n-m} is transversal to $\tau(p')$, p' being the point where $\pi^{n-m}(p)$ intersects P^m, and (2) π^{n-m} meets S in an analytic $(n-m-1)$-sphere $S^*(p, \pi^{n-m})$ contained in

[12] The restrictions are obtained by conditions on Φ', ω, and their gradients.

$R(P^m)$. Let $Q^*(p, \pi)$ be the part of π^{n-m} inside $S^*(p, \pi^{n-m})$. The following results are then proved. (1) The vector function $g(p, \pi)$ representing the center of mass of $Q^*(p, \pi)$ is analytic. (2) If $\pi^*(p)$ is a sufficiently close anlytic approximation to $\pi^{n-m}(p)$ through first order derivatives, then the locus of the centers of mass $g(p, \pi^*(p))$ is an analytic manifold, M^*, homeomorphic to P^m. This manifold can be made arbitrarily close to P^m.

If it is desired merely to make P^m, and hence M, differentiable to any given order $r \epsilon (1, 2, \cdots, \infty)$, this can be done as follows. Construct $\pi^{n-m}(p)$ by the recurrency of §§7 and 8 so that it is of class C^r in $R(P^m)$. On each $\tau(p)$, let there be introduced a fixed rectangular cartesian coordinate system, with its domain restricted to the part of $\tau(p)$ inside $R(P^m)$. If p' denote a point on such a domain, then the plane $\pi^{n-m}(p')$ [see Hypothesis IV] meets P^m in a single point near p'. This affords a mapping which carries the coordinate system from $\tau(p)$ onto P^m. In terms of coordinate systems thus defined on P^m, the latter is of class C^r since the transformation between any two such systems agrees with the correspondence established by $\pi^{n-m}(p)$ in $R(P^m)$ between rectangular cartesian systems and two m-planes $\tau(p_0)$ and $\tau(p_1)$. This method enables us to dispense with the hypothesis that there be an upper bound to the number of cells in a star on P^m [see footnote 4]. For we can apply the above argument over a sequence of finite subcomplexes each containing the preceding and, in the limit, covering M. This makes the entire manifold M of class C^r. It then remains only to apply DM, Theorem 1.

The existence of differentiable approximations to polyhedral manifolds was prematurely asserted by the writer[13], who is indebted to Hassler Whitney for calling his attention to the incompleteness of his work. The results of the present paper include the theorem of the abstract only for the case $m = 3$. This is the strongest such theorem which the writer has thus far proved.

10. The Brouwer nature of M ($m \leqq 3$). LEMMA. *Let S be any star of an $(m - k)$-cell on a P^m. Then, if $k \leqq 3$, it is possible to map S by a piecewise linear homeomorphism into an E^m.*

PROOF. In view of §4, Lemma, and the work in §6, it is sufficient to show that any triangulation (σ) of a $(k - 1)$-sphere, S^{k-1}, can be mapped homeomorphically into a geodesic triangulation of a $(k - 1)$-sphere. The proof is trivial for $k < 3$, so we restrict ourselves to the case $k = 3$. We employ a recurrency with the following basic hypothesis.

HYPOTHESIS. For some value $j > 0$, a subcomplex $(\sigma)_j$ of (σ), consisting of j 2-cells with their boundaries, has been mapped topologically into a geodesic complex (τ_j) on S^2 so that the part of S^2 not covered by $(\tau)_j$ is the sum of a finite number of convex regions, the closure of each of which is a subset of an open hemisphere and none of which has three of its boundary vertices on a

[13] Bulletin of the American Mathematical Society, vol. XL (1934), Abstract 67.

great circle. If β is the boundary of any one of these regions, ρ, then the image, β', of β bounds a subcomplex $(\sigma)^*$ of (σ) containing no 2-cell of $(\sigma)_j$.

For the initial step of the proof, consider any vertex, P, of (σ). It is then a simple matter to map the closure of the star $S(P)$, relative to (σ), into a complex $(\tau)_j$ so that the conditions of the hypothesis are fulfilled for j equal to the number of 2-cells on $S(P)$.

To define the general step, using the notation of the hypothesis, let P be a vertex on β' and let $S(P)$ denote its star relative to the subcomplex $(\sigma)^*$ of (σ). There is no difficulty in mapping $S(P)$ onto $(\beta + \rho)$ so as to secure the conditions of the hypothesis for a larger value of j. After a certain finite number of such steps, the mapping will be complete.

COROLLARY. *Every P^m $(m \leqq 3)$, and hence every triangulated m-manifold $(m \leqq 3)$, is a Brouwer manifold.*

This is the special case of the lemma in which $m \leqq 3$ and $k = m$.

11. Establishment of Theorem III. The case $m = 4$. LEMMA 11.1. *It is possible, for any P^m in general position, to define $\pi^{n-m}(p)$ on the regions $\mathfrak{N}(\gamma^j)$ $(j \geqq m - 3)$ so that $\pi^{n-m}(p)$ will be continuous in p and will be transversal to P^m at p.*

PROOF. We will assume $m \geqq 3$, so that there will exist stars $S(s^{m-3})$. The lower-dimensional cases require only part of the following argument. By §10, Lemma, every $S(s^{m-3})$ is a Brouwer star. Hence, given a point p_0 on any γ^{m-3}, we can define a plane $\pi^{n-m}(p_0)$ transversal to P^m at p_0. We can then extend the definition as p ranges over the γ^{m-3} containing p_0 by the requirement $\pi^{n-m}(p) \parallel \pi^{n-m}(p_0)$. Since the γ^{m-3} are bounded away from one another, this can be done independently for each of them. It is then possible to proceed with the recurrency of §§7 and 8, starting with the second part of the step $j = m - 3$. In order to apply the condition of Theorem V, as in §8, to the remaining steps, we have to note that (1) in any $T(\tau^1)$, every $(m - 3)$-sphere bounds[14] a cell, (2) in $T(\tau^0)$, every $(m - 2)$-sphere bounds a cell, and (3) every $(m - 1)$-sphere bounds a cell in $T(\tau^{-1})$ [see footnote 9].

The case $m \leqq 3$ gives the following result.

COROLLARY. *Every P^m $(m \leqq 3)$ can be put in normal position in some E^n.*

From this corollary and Theorem II, we deduce Theorem III.

LEMMA 11.2. *If every $T(\tau^2)$ is connected, then every Brouwer 4-manifold can be put in normal position.*

PROOF. The connectedness of the $T(\tau^2)$ being assumed, we have only to note that the $T(\tau^1)$ are simply connected[14] and that the higher-dimensional connectivities of $T(\tau^0)$ and of $T(\tau^{-1})$ satisfy the conditions of Theorem V.

Thus the normal position problem for Brouwer 4-manifolds reduces to the following.

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[14] For, $T(\tau^1)$ is a μ-cell, where $\mu + 2$ is the number of vertices of τ^1.

Deformation problem. Given two geodesic triangulations, (σ) and (τ), of a 2-sphere, which correspond under an orientation-preserving homeomorphism, does there exist a continuously varying geodesic triangulation $(\sigma)_t$ $(0 \leqq t \leqq 1)$, such that $(\sigma)_0 = (\sigma)$, $(\sigma)_1 = (\tau)$, and $(\sigma)_t$ is always homeomorphic to $(\sigma)_0$?

Some of Tietze's work[15] has a bearing on this deformation problem, but the writer has not succeeded in obtaining a solution save in a few special cases.

(*A*) *It follows from Theorem II and Lemma* 11.2 *that every* 4-*dimensional Brouwer manifold can be made into an analytic Riemannian manifold if the deformation problem has an affirmative solution.*

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[15] Renditiconti del circolo matematico di Palermo, vol. 38 (1914), pp. 247–304, especially *Satz IV* on p. 280.

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ON C^1-COMPLEXES

By J. H. C. Whitehead

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1. This paper is supplementary to S. S. Cairns' work[1] on the triangulation of, and polyhedral approximations to manifolds of class C^1. Its aim is to provide a foundation for theorems[2] which involve both differential geometry and the theory of combinatorial equivalence.[3] Theorem 8, for example, states that two C^1-triangulations of any manifold of class[4] C^1 are combinatorially equivalent. Thus a manifold of class C^1 is like a recti-linear polyhedron in that it has a preferred class of combinatorially equivalent triangulations. This theorem depends on the definition of a C^1-complex given in §2 below, and does not apply, for example, to the algebraic complexes first considered by B. L. van der Waerden.[5] For though two "algebraic triangulations" of the same space have a common algebraic sub-division, it is not certain that an algebraic triangulation of a recti-linear n-simplex is combinatorially equivalent to an n-simplex. The C^1-complexes, though more closely allied to, also differ essentially from the complexes considered by Cairns. Therefore we do not use Cairns' results, though the main ideas in many of our theorems are due to him.

We shall use R^n to stand for n-dimensional Euclidean space, and it is to be understood that R^n is Euclidean not only in its topology, but also in the sense of metric geometry. By a complex K we shall always mean a recti-linear, simplicial complex in Euclidean space, and $\dot{K}$ will stand for the mod 2 boundary of K. We shall denote a recti-linear, simplicial subdivision of K, but not necessarily a normal sub-division, by K', and if K_0 is any sub-complex of K, then K_0' will be the sub-complex of K' covering K_0. By a simplex we shall always mean a closed simplex (i.e. a simplex with its boundary), and we shall use the letters A, B to stand for recti-linear simplexes. By an *isomorphism* $t(K_1) = K_2$ we shall mean a homeomorphism of K_1 on K_2 which maps each simplex of K_1 on the whole of one, and only one, simplex of K_2, and which is linear throughout each simplex in K_1. In §§2 and 3, theorem 5 excepted, K will always stand for a finite complex. We shall use the summation con-

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[1] Annals of Math., 35(1934), 579–87 (triangulation); 37(1936), 409–15 (approximations).

[2] See, for example, J. H. C. Whitehead, Annals of Math. 41(1940), 825–832.

[3] Here we may take as a definition: two simplicial complexes K_1 and K_2, finite or infinite, are combinatorially equivalent if, and only if, recti-linear models of K_1 and K_2 have isomorphic recti-linear sub-divisions.

[4] See, for example, O. Veblen and J. H. C. Whitehead, *The Foundations of Differential Geometry*, Cambridge (1932), Chap VI. By an n-dimensional manifold we shall always mean one which is covered by a countable set of open n-cells.

[5] Math. Ann., 102(1929), 337–62.

vention in analytical formulae, with the additional convention that Roman indices take on the values $1, \cdots, n$.

2. A map $f(U) \subset R^n$, of a region $U \subset R^k$ (possibly $U \subset R^k \subset R^m$, where $m > k$), is said to be of class C^1, or a C^1-map, if and only if, it is given by equation of the form

$$y^i = f^i(x^1, \cdots, x^k) = f^i(x), \tag{2.1}$$

where $x^1, \cdots, x^k$ and $y^1, \cdots, y^n$ are rectangular Cartesian coordinates for R^k and for R^n, and the functions $f^i(x)$ have continuous derivatives at each point of U. The map f will be described as *non-degenerate* if, and only if, the Jacobian matrix of the transformation (2.1) is of rank k at each point of U. A map $f(A) \subset R^n$, of a k-simplex $A \subset R^k$ will be described as of class C^1 (non-degenerate) if, and only if, it can be extended throughout some open set in R^k, containing A, in which it is of class C^1 (non-degenerate).

Let $f_\alpha(A) \subset R^n$ ($\alpha = 1, 2$) be two C^1-maps of a simplex A, given by equations of the form (2.1). We shall describe f_2 as an (ϵ, ρ)-approximation to f_1 if, and only if,

$$\|f_2 - f_1\| \leqq \epsilon \quad \text{and} \quad \|df_2 - df_1\| \leqq \rho \|df_1\|$$

for each $x \in A$ and every vector dx, where

$$df^i_\alpha = \frac{\partial f^i_\alpha}{\partial x^\lambda} dx^\lambda \qquad (\lambda = 1, \cdots, k)$$

and $\|y_2 - y_1\| = \{(y^i_2 - y^i_1)(y^i_2 - y^i_1)\}^{\frac{1}{2}}$, $\|y\| = (y^iy^i)^{\frac{1}{2}}$. When we are only interested in $\|df_2 - df_1\|$, or in $\|f_2 - f_1\|$, we may refer to f_2 as an (∞, ρ)-, or as an (ϵ, ∞)-approximation to f_1. Let a be the origin of the coordinates y and let p_1 and p_2 be the extremities of the vectors df_1 and df_2, situated at a. If f_2 is an (∞, ρ)-approximation to f_1, it follows from the geometry of the triangle ap_1p_2 that $ap_2 \leqq ap_1 + p_1p_2 \leqq (1 + \rho)ap_1$, and $ap_1 \leqq ap_2 + p_2p_1 \leqq ap_2 + \rho ap_1$, whence $(1 - \rho)ap_1 \leqq ap_2$. Therefore, if $\rho \leqq \frac{1}{2}$, we have $ap_1 \leqq 2ap_2$ and f_1 is an $(\infty, 2\rho)$-approximation to f_2. If f_2 is an (∞, ρ_1)-approximation to f_1, f_3 an (∞, ρ_2)-approximation to f_2, and if ap_3 is the vector df_3, then,

$$p_1p_3 \leqq p_1p_2 + p_2p_3 \leqq \rho_1 ap_1 + \rho_2 ap_2 \leqq (\rho_1 + \rho_2 + \rho_1\rho_2)ap_1,$$

since $ap_2 \leqq (1 + \rho_1)ap_1$. Therefore f_3 is an (∞, ρ_3)-approximation to f_1, where $\rho_3 = \rho_1 + \rho_2 + \rho_1\rho_2$. Notice that $\rho_3 < 3\rho_1$ if $\rho_1 = \rho_2 < 1$. Combining these with the familiar relations for (ϵ, ∞)-approximations, we see that, if f_2 is an (ϵ_1, ρ_1)-approximation to f_1, where $\rho_1 \leqq \frac{1}{2}$, and f_3 is an (ϵ_2, ρ_2)-approximation to f_2, then f_1 is an $(\epsilon_1, 2\rho_1)$-approximation to f_2 and f_3 an $(\epsilon_1 + \epsilon_2, \rho_1 + \rho_2 + \rho_1\rho_2)$-approximation to f_1. Notice also that an (∞, ρ)-approximation, f_2, to a non-degenerate map, f_1, is itself non-degenerate if $\rho < 1$. For to say that f is non-degenerate is to say that $df \neq 0$ if $dx \neq 0$, and $\|df_2\| \geqq (1 - \rho)\|df_1\|$.

LEMMA 1. *If f_2 is a non-degenerate (∞, ρ)-approximation to a non-degenerate map f_1, then the angle*[6] *between the vectors df_1 and df_2 does not exceed $\pi\rho$.*

If $\rho \geqq 1$ or if $\rho = 0$ this is trivial. So we assume $0 < \rho < 1$. Then, with the same notation as before, if the lengths ap_1 and p_1p_2 are fixed, the angle θ = angle p_1ap_2 is greatest when ap_2 touches a circle of radius p_1p_2 and center p_1, in which case p_1p_2 is perpendicular to ap_2. Therefore $\sin \theta \leqq p_1p_2/ap_1 \leqq \rho$, whence $\theta \leqq \frac{1}{2}\pi \sin \theta < \pi\rho$ and the lemma is established.

By the *radius* $r(A)$, of a simplex $A \subset R^n$ we shall mean the distance from its centroid to its boundary. Let $f_\alpha(A) \subset R^n$ $(\alpha = 1, 2)$ be two linear, non-degenerate maps of a k-simplex $A \subset R^k$, let ϵ be the maximum of $\| f_2(x) - f_1(x) \|$ as x varies over A, and let $r = r\{f_1(A)\}$.

LEMMA 2. *Under these conditions $f_2(A)$ is an $(\epsilon, 2\epsilon/r)$-approximation to $f_1(A)$.*

By definition, $f_2(A)$ is an (ϵ, ∞)-approximation to $f_1(A)$, and we have only to prove that $\| df_2 - df_1 \| \leqq \rho \| df_1 \|$, where $\rho = 2\epsilon/r$. Since f_1 and f_2 are linear, $df_2 - df_1$ does not depend on the point $x \in A$, but only on the vector dx. Therefore we may take $x = \bar{x}$, the centroid of A, and we take $\bar{x} + dx \in \dot{A}$. Then $f_1(\bar{x})$ is the centroid of $f_1(A)$, since f_1 is linear, and $f_1 + df_1 = f_1(\bar{x} + dx) \in \{f_1(A)\}$; whence $\| df_1 \| \geqq r$. Therefore

$$\begin{aligned} \| df_2 - df_1 \| &= \| (f_2 + df_2) - (f_1 + df_1) - (f_2 - f_1) \| \\ &= \| \{f_2(\bar{x} + dx) - f_1(\bar{x} + dx)\} - \{f_2(\bar{x}) - f_1(\bar{x})\} \| \\ &\leqq \| f_2(\bar{x} + dx) - f_1(\bar{x} + dx) \| + \| f_2(\bar{x}) - f_1(\bar{x}) \| \leqq 2\epsilon \leqq \frac{2\epsilon}{r} \| df_1 \| \end{aligned}$$

and the lemma is established.

Let A be a k-simplex in R^k and let $f(A) \subset R^n$ be a non-degenerate C^1-map, given by equations of the form (2.1). If b is any point in A these equations may be rewritten as

$$y^i - c^i = a^i_\lambda(x^\lambda - b^\lambda) + \gamma^i(x), \tag{2.2}$$

where $c = f(b)$ and a^i_λ are the derivatives $\partial f^i/\partial x^\lambda$, calculated for $x = b$. The image of A in the linear transformation F_b, given by

$$y^i - c^i = a^i_\lambda(x^\lambda - b^\lambda), \tag{2.3}$$

will be called the *tangent simplex* to $f(A)$ at the point c. Since the derivatives $\partial f^i/\partial x^\lambda$ are continuous, and therefore uniformly continuous in the compact set A, it is an obvious consequence of lemma 2 that, given $\epsilon, \rho > 0$, there is a $\delta > 0$ such that F_b is an (ϵ, ρ)-approximation to $F_{b'}(A)$ provided $\| b' - b \| \leqq \delta$. Since, at the point $x = b'$, $df = dF_{b'}$, we have:

LEMMA 3. *Given $\epsilon, \rho > 0$, there is a positive δ such that F_b is an (ϵ, ρ)-approximation to f throughout the sub-set of A given by $\| x - b \| \leqq \delta$, for any $b \in A$.*

By the relative thickness,[7] $\tau(B)$, of a simplex $B \subset R^m$, we shall mean r/l,

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[6] By the angle between two vectors we mean the positive angle which does not exceed π.

[7] Cf. Cairns' definition of a θ-set (Triangulation, p. 583).

where $r = r(B)$ and l is the diameter,[8] $l(B)$, of B. Let $f(A) \subset R^n$ be a non-degenerate C^1-map of a k-simplex $A \subset R^k$, let $b_0, \cdots, b_p$ be any points in A and let B be the simplex $b_0 \cdots b_p$. Let $L(B) \subset R^n$ be the linear map of B such that $L(b_\alpha) = f(b_\alpha)$ $(\alpha = 0, \cdots, p)$.

LEMMA[9] 4. *Given $\epsilon, \rho, \sigma > 0$, there is a positive δ such that, if $r(B) \geqq \sigma$ and $l(B) \leqq \delta$, then $L(B)$ is an (ϵ, ρ)-approximation to $f(B)$.*

By lemma 3 there is a positive δ_1 such that $F_b(B)$ is an $(\epsilon/2, \rho/3)$-approximation to $f(B)$ for any $b \in B$, provided $l(B) \leqq \delta_1$. Therefore, assuming, as we obviously may, that $\rho < 1$, the lemma will follow, with $\delta = \min(\delta_1, \delta_2)$, if there is a positive δ_2 such that $L(B)$ is an $(\epsilon/2, \rho/3)$- approximation to $F_b(B)$ for any $b \in B$, provided $l(B) \leqq \delta_2$. The transformations $F_b(B)$ and $L(B)$ are given by

$$y^i - c^i = a^i_\lambda(b^\lambda_\alpha - b^\lambda)t^\alpha$$

$$y^i - c^i = a^i_\lambda(b^\lambda_\alpha - b^\lambda)t^\alpha + \gamma^i(b_\alpha)t^\alpha \qquad (\alpha = 0, \cdots, p),$$

where $0 \leqq t^\alpha \leqq 1$, $t^0 + \cdots + t^p = 1$, $b^1_\alpha, \cdots, b^k_\alpha$ are the coordinates of the vertex b_α, and a^i_λ and $\gamma^i(x)$ mean the same as in (2.2). Since the derivatives of $\gamma^i(x)$ vanish when $x = b$ there is a positive $\delta(\eta)$, such that $|\gamma^i(x)| \leqq \eta \| x - b \|$ provided $\| x - b \| \leqq \delta(\eta)$, for a given $\eta > 0$. If $l = l(B) = \delta(\eta)$ and if $b \in B$ it follows that $|\gamma^i(b_\alpha)| \leqq \eta l$. Since $t^\alpha \geqq 0$ and $t^0 + \cdots + t^p = 1$ we have $|\gamma^i(b_\alpha)t^\alpha| \leqq \eta l$, whence

$$\| L(x) - F_b(x) \| = \| \gamma(b_\alpha)t^\alpha \| \leqq \eta l n^{\frac{1}{2}}$$

for any point $x = b_\alpha t^\alpha$ in B. It follows from lemma 2 that $L(B)$ is an $(\eta l n^{\frac{1}{2}}, 2\eta l n^{\frac{1}{2}}/r_b)$-approximation to $F_b(B)$, where $r_b = r\{F_b(B)\}$.

Now let $x^1, \cdots, x^k$ in (2.3), be rectangular coordinates for $R^k \supset A$, and let $\zeta_1(b)$ be the smallest root of the equation

$$|a^i_\lambda a^i_\mu - \zeta\delta_{\lambda\mu}| = 0. \tag{2.4}$$

Then $\zeta_1(b)$ is a continuous function of b, and is positive since f is non-degenerate. Therefore $\zeta_1(b)$ has a positive lower bound, ω^2, as b varies in the compact set A, and

$$\| dF_b \| = (a^i_\lambda a^i_\mu \, dx^\lambda \, dx^\mu)^{\frac{1}{2}} \geqq \omega \| dx \|$$

for any $b \in A$. Therefore $r_b \geqq \omega r$, where $r = r(B)$, and $l/r_b \leqq l/\omega r \leqq 1/\omega\sigma$, since $r/l \geqq \sigma$, whence

$$2\eta l n^{\frac{1}{2}}/r_b \leqq 2\eta n^{\frac{1}{2}}/\omega\sigma.$$

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[8] $l(B)$ is the length of the longest side of B (P. Alexandroff and H. Hopf, *Topologie*, Berlin (1935), 607).

[9] Cf. Cairns (Approximations, §4).

Also $l \leqq l(A) = l_1$, say. Therefore $L(B)$ is an $(\epsilon/2, \rho/3)$-approximation to $F_b(B)$ provided $l \leqq \delta_2 = \delta(\eta)$, where

$$\eta = \min\,(\epsilon/2l_1n^{\frac{1}{2}}, \qquad \rho\omega\sigma/6n^{\frac{1}{2}}),$$

and the lemma is established.

By a *(δ, σ)-subdivision* of $K \subset R^m$ we shall mean a subdivision K', such that $l(B) \leqq \delta$, $\tau(B) \geqq \sigma$, where B is any simplex in K' and, as before, $l(B)$ and $\tau(B)$ stand for the diameter and the relative thickness of B.

LEMMA[10] 5. *There is a (δ, σ)-subdivision of K for an arbitrary $\delta > 0$ and some $\sigma > 0$, which does not depend on the choice of δ.*

Let the equations (2.3) now represent an arbitrary, non-singular, linear transformation $F(R^k) \subset R^n$. If ζ_1 and ζ_k are the smallest and greatest roots of the equation (2.4) we have

$$\zeta_1^{\frac{1}{2}}\, \|\, dx \,\| \leqq \|\, dy \,\| \leqq \zeta_k^{\frac{1}{2}}\, \|\, dx \,\|.$$

If B is any simplex in R^k and $C = F(B)$ it follows that $\tau(B) \geqq \kappa\tau(C)$, where $\kappa = (\zeta_1/\zeta_k)^{\frac{1}{2}}$, and $\kappa > 0$ since F is non-singular. Therefore, if $K = F(K_1)$, where $K_1 \subset R^N$ and F is an isomorphism, and if K_1' is any subdivision of K_1, there are constants ω_0, κ_0, such that $l(B) \leqq \omega_0 l(B_1)$, $\tau(B) \geqq \kappa_0\tau(B_1)$, where B_1 is any simplex in K_1' and $B = F(B_1)$. Therefore we may replace K by an isomorphic complex in R^N, and shall assume it to be a sub-complex of the simplex Δ_1, whose vertices have rectangular Cartesian coordinates $(0, \cdots, 0)$, $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$, where $N + 1$ is the number of vertices in K. Let P be the polyhedral complex consisting of the convex cells into which R^N is divided by the hyperplanes

$$y^\beta = k \qquad (\beta = 0, \cdots, N;\, k = 0, \pm 1, \pm 2, \cdots),$$

where $y^1, \cdots, y^N$ are the coordinates for R^N and $y^0 = y^1 + \cdots + y^N$. Then P contains a sub-complex covering the simplex Δ_q, whose vertices are the points $(0, \cdots, 0)$, $(q, 0, \cdots, 0), \cdots, (0, \cdots, 0, q)$, for any integral value of q. The complex P also contains a sub-complex Q, which covers the hyper-cube given by $0 \leqq y^\lambda \leqq 1$ $(\lambda = 1, \cdots, N)$, and each cell in P is congruent to a cell in Q under the group of translations. Let P' and Q' be the complexes obtained from P and Q by a normal subdivision, the new vertices being placed at the centroids of the corresponding cells. Then each simplex in P' is congruent to some simplex in Q' and its relative thickness is therefore at least σ, where σ is the minimum relative thickness of the simplexes in Q'. Let E_q be the sub-complex of P' covering Δ_q and let E_1 be the image of E_q in the transformation given by $\bar{y}^\lambda = y^\lambda/q$. Then E_1 is a subdivision of Δ_1 and so contains a sub-complex K', which is a subdivision of K. The relative thickness of each simplex in K' is at least σ, since the relative thickness is an invariant of the

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[10] Cf. Cairns (Triangulation, p. 585).

similarity group, and its diameter is less than $\frac{1}{q}N^{\frac{1}{2}}$. Taking $q \geqq \frac{1}{\delta}N^{\frac{1}{2}}$ the lemma is established.

By a *C^1-map, $f(K) \subset R^n$*, or a *map of class C^1*, we shall mean a map which is of class C^1 throughout each simplex in K. The map $f(K)$ will be described as non-degenerate if, and only if, it is non-degenerate throughout each simplex. We shall also describe $f(K)$ as a C^1-complex, or a complex of class C^1, and as a non-degenerate complex if, and only if, the map f is non-degenerate. By an *(ϵ, ρ)-approximation* to $f(K)$, we shall mean a C^1-map $f'(K') \subset R^n$, where K' is any sub-division of K, such that $f'(A)$ is an (ϵ, ρ)-approximation to $f(A)$ throughout each simplex $A \subset K'$. We shall use $L_f(K')$ to denote the map which is linear (possibly degenerate) throughout each simplex of K' and coincides with f at the vertices of K'. Notice that $L_f(K')$ is not, in general, the image of K' in $L_f(K)$.

THEOREM 1. *Given a non-degenerate C^1-complex $f(K) \subset R^n$, and $\epsilon, \rho, \sigma > 0$, there is a positive δ such that $L_f(K')$ is an (ϵ, ρ)-approximation to $f(K)$, where K' is any (δ, σ)-sub-division of K.*

By lemma 4 there is a $\delta(A) > 0$ such that, if K' is any $\{\delta(A), \sigma\}$-subdivision of K, then $L_f(A')$ is an (ϵ, ρ)-approximation to $f(A)$, where A' is the subcomplex of K' covering $A \subset K$. Taking $\delta = \min \delta(A)$, for any $A \subset K$, the theorem follows.

Let K_1 be a sub-complex of a given complex K and let K_1' be any subdivision of K_1. By an *extension* of K_1' throughout K we shall mean a subdivision K' of K, which coincides with K_1' in K_1. Let $f(K) \subset R^n$ be a non-degenerate C^1-map and let $f_1(K_1') \subset R^n$ be an (ϵ_1, ρ_1)-approximation to $f(K_1)$. By an *(ϵ, ρ)-extension* of $f_1(K_1')$ throughout K we shall mean an (ϵ, ρ)-approximation, $f'(K') \subset R^n$, to $f(K)$, which coincides with f_1 in K_1', where K' is an extension of the subdivision K_1'.

THEOREM 2. *Given a non-degenerate C^1-map, $f(K) \subset R^n$, a sub-complex $K_1 \subset K$ and $\epsilon, \rho > 0$, there are positive numbers ϵ_1, ρ_1, such that any (ϵ_1, ρ_1)-approximation to $f(K_1)$ has an (ϵ, ρ)-extension throughout K.*

This will follow from an obvious induction on the number of simplexes in $Cl(K - K_1)$, the closure of $K - K_1$, when we have proved it in case $K = A$, a single simplex, and $K_1 = \dot{A}$. Let a be the centroid of A, let x_0 be the mid point of the segment ax_1, for any $x_1 \epsilon \dot{A}$, and let A_0 be the simplex bounded by the locus of x_0 as x_1 describes $\dot{A}$. Let $\dot{A}'$ be any subdivision of $\dot{A}$ and let P be the polyhedral complex consisting of the convex cells $B \times x_0x_1$, swept out by the segment x_0x_1 as x_1 varies over the simplexes $B \subset \dot{A}'$. Let x_t be the point on x_0x_1 such that $x_0x_t:x_tx_1 = t:(1-t)$ $(0 \leqq t \leqq 1)$ and let $x_1^1, \cdots, x_1^k, t$ be taken as coordinates for $B \times x_0x_1$, where $x_1^1, \cdots, x_1^k$ are Cartesian coordinates for any $B \subset \dot{A}'$. Let Δ be the maximum and δ the minimum attained by $\|df\|$ for any $x_t \epsilon P$ and any vector $(dx_1^1, \cdots, dx_1^k, dt)$ whose length is unity in terms of a Euclidean metric for A.

Now let $f_1(\dot{A}') \subset R^n$ be an (ϵ_1, ρ_1)-approximation to $f(\dot{A})$, and, treating $f(x_1)$

and $f_1(x_1)$ as vectors in R^n, let $\gamma(x_1) = f_1(x_1) - f(x_1)$. Then $\|\gamma\| \leqq \epsilon_1$ and, if dx_1 is a unit vector, $\|d\gamma\| \leqq \rho_1 \|df\| \leqq \rho_1\Delta$, whence

$$(2.5) \qquad \begin{cases} |\gamma^i| \leqq \|\gamma\| \leqq \epsilon_1 \\ |d\gamma^i| \leqq \|d\gamma\| \leqq \rho_1\Delta, \end{cases}$$

where the index i refers to some rectangular Cartesian coordinate system for R^n. Let $f'(P)$ be the C^1-map given, in vector notation, by

$$f'(x_t) = f(x_t) + t\gamma(x_1) \qquad (0 \leqq t \leqq 1; x_1 \epsilon \dot{A}').$$

Then

$$\|f' - f\| = t\|\gamma\| \leqq \epsilon_1 \quad \text{and} \quad \|df' - df\| = \|t\,d\gamma + \gamma\,dt\|.$$

Taking (dx_1, dt) to be a unit vector, in which case $|dt|$ is bounded, it follows from (2.5) and the continuity of the function $\|y\|$ that there are positive numbers ϵ_1 and ρ_1 such that $\|df' - df\| \leqq \rho\delta \leqq \rho\|df\|$. If $\epsilon_1 \leqq \epsilon$ we have also $\|f' - f\| \leqq \epsilon$ and $f'(P)$ is an (ϵ, ρ)-approximation to $f(P)$. Finally we take $f' = f$ in A_0 and extend the sub-division $\dot{A}'$ by starring A_0 and each of the cells $B \times x_0x_1$, leaving $\dot{A}'$ untouched. The result is an (ϵ, ρ)-extension of f_1 throughout A, and the theorem is established.

Let K_2 be the complex consisting of all the simplexes in K which do not meet K_1. As a corollary to theorem 2, replacing $f_1(K_1')$ by $f_2(K_1' + K_2)$ with $f_2 = f_1$ in K_1', $f_2 = f$ in K_2, we have the addendum:

ADDENDUM. *The extension $f'(K')$, referred to in theorem 2, may be chosen so that the subdivision K' leaves K_2 unaltered and $f' = f$ in K_2.*

If B is any simplex in K we shall use $N(B, K)$ to stand for the *stellar neighbourhood* of B in K, consisting of all the simplexes $AB \subset K$, where AB is the join[11] of A and B. If b is an internal point of B we shall also describe $N(B, K)$ as the *stellar neighbourhood*,[12] $N(b, K)$, of b. If $f(K) \subset R^n$ is any C^1-complex, the recti-linear complex in R^n which consists of the tangent simplexes at $f(b)$ to the simplexes in $f\{N(b, K)\}$, will be called the *tangent star at $f(b)$ to $f(K)$*. Thus the tangent star is the image of $N(b, K)$ in a simplicial transformation F_b, which coincides with the transformation $F_b(AB)$, defined by (2.3), throughout each of the simplexes AB, where b is internal to B. By a *non-singular C^1-complex, or map*, $f(K)$ we shall mean a C^1-complex such that

1. *f is $(1 - 1)$ throughout K,*
2. *F_b is $(1 - 1)$ throughout $N(b, K)$ for each point $b \epsilon K$.*

It follows from the second of these conditions that a non-singular map is non-degenerate.

THEOREM 3_ρ. *To any non-singular C^1-complex $f(K) \subset R^n$ correspond positive numbers ϵ, ρ, such that any (ϵ, ρ)-approximation to $f(K)$ is non-singular.*

According to a previous observation, any (ϵ, ρ)-approximation to f is non-

[11] Here we allow A to be 1, the empty simplex, in which case $AB = B$.

[12] In general $N(B, K) \subset$ but $\neq N(b, K)$ if $b \epsilon \dot{B}$.

degenerate if $\rho < 1$. Thus, taking $\rho < 1$, we may confine ourselves to non-degenerate approximations. On this understanding we shall prove a similar theorem with less restrictive hypotheses. A non-degenerate map $f'(K') \subset R^n$ will be called an $| \epsilon, \alpha |$-approximation to $f(K)$ if, and only if, $\| f' - f \| \leqq \epsilon$ and the angle between df and df' is at most α, for each $x \in K'$ and non-zero vector dx, in any simplex of K'. Notice that this relation is symmetric between f and f_1, and if f_1 is an $| \epsilon_1, \alpha_1 |$-approximation to f and f_2 an $| \epsilon_2, \alpha_2 |$-approximation to f_1, then f_2 is an $| \epsilon_1 + \epsilon_2, \alpha_1 + \alpha_2 |$-approximation to f. By lemma 1 an (ϵ, ρ)-approximation is an $| \epsilon, \pi\rho |$-approximation, but an example of the form $y = x + \epsilon \sin \lambda x$ $(\epsilon, \lambda > 0; 0 \leqq x \leqq \pi/2\lambda)$ shows that an $| \epsilon, 0 |$-approximation need not be an (ϵ, ρ)-approximation for any given ρ. Our theorem is:

THEOREM 3_α. *To any non-singular C^1-complex $f(K) \subset R^n$ correspond positive numbers ϵ, α such that any $| \epsilon, \alpha |$-approximation to $f(K)$ is non-singular.*

First consider the special case in which $K = x_1x_0 + x_0x_2$, where x_0x_λ $(\lambda = 1, 2)$ are linear segments with no common point other than x_0, and $f(x_1x_0 + x_0x_2) = y_1y_0 + y_0y_2$ is linear throughout each of x_0x_λ. In this case any (non-degenerate) $| \infty, \theta |$-approximation $f'(K')$, to $f(K)$, is $(1 - 1)$ provided $2\theta <$ angle $y_1y_0y_2$. For let R^2 be the plane containing y_0, y_1 and y_2, or any plane through these points if they are collinear, and let $l \subset R^2$ be the external bisector of the angle $y_1y_0y_2$, or the line $y_1y_0y_2$ if these points are collinear. Then the inclination to y_0y_λ $(\lambda = 1$ or $2)$ of any direction in R^n perpendicular to l, is at least $\frac{1}{2}$ angle $y_1y_0y_2$. If $2\theta <$ angle $y_1y_0y_2$ it follows that the vector df' is never perpendicular to l, and its orthogonal projection on l points away from y_0. Therefore the orthogonal projection of $f'(K')$ on l is a non-singular image of K', whence f' is $(1 - 1)$.

Let A_1B and A_2B be simplexes in $R^m (A_\lambda \neq 1)$, let b_0 be the centroid of B, and let $\theta_0(A_1, A_2, B)$ be the minimum attained by the angle a_1b_0p for[13] $a_1 \in A_1$, $p \in A_2\dot{B}$, and let

$$\theta(A_1, A_2, B) = \min \{\theta_0(A_1, A_2, B), \pi/2\}.$$

It follows from a standard type of argument that $\theta_0(A_1, A_2, B)$, and hence $\theta(A_1, A_2, B)$ vary continuously with the vertices of A_1, A_2 and B, provided the simplexes A_1B and A_2B remain non-degenerate (and under less stringent conditions). Let $x_\lambda \in A_\lambda B - B$ $(\lambda = 1, 2)$, let a_1 be the point in A_1 such that the simplex a_1B contains x_1, and let the line through x_1 parallel to a_1b_0 meet B in b. Then the line through b_0 parallel to bx_2 meets $A_2\dot{B}$ in p, say, and angle $x_1bx_2 =$ angle $a_1b_0p \geqq \theta(A_1, A_2, B)$. Notice that the construction for b, given x_1, is affine and so invariant under a linear transformation. We shall call b the A_1-projection of x_1 in B.

If A_1B and A_2B do not meet except in B we have $\theta = \theta(A_1, A_2, B) > 0$, and I say that $\| x_\lambda - b \| \leqq \| x_2 - x_1 \| \operatorname{cosec} \theta$, where x_λ and b mean the same

[13] $A_2\dot{B} = A_2$ if B is a 0-simplex.

as before. For if angle $x_1bx_2 \geqq \pi/2$ we have $\| x_\lambda - b \| < \| x_2 - x_1 \| \leqq \| x_2 - x_1 \|$ cosec θ, and if angle $x_1bx_2 < \pi/2$ we have

$$\| x_\lambda - b \| \leqq \| x_2 - x_1 \| \operatorname{cosec} (x_1bx_2) \leqq \| x_2 - x_1 \| \operatorname{cosec} \theta,$$

since $\| x_\lambda - b \|$ does not exceed the diameter of the circle through x_1, x_2 and b.

Now let $A_\lambda B \subset K$, $A_\lambda \neq 1$ ($\lambda = 1, 2$; $A_1 \cdot A_2 = 0$) and let

$$\theta(p) = \theta\{F_p(A_1), F_p(A_2), F_p(B)\},$$

where $p \in B$. Since f is non-singular, $\theta(p)$ is a positive, continuous function of $p \in B$ and so attains a positive minimum. Let 7α be the least of these minima, calculated for every pair of simplexes in K which have a common point though neither is contained in the other, and let c be the greatest of the numbers cosec $\theta(A_1, A_2, B)$. Let $f'(K')$ be an $| \infty, \alpha |$-approximation to $f(K)$. We first show that the map F'_p is non-singular[14] for each $p \in K$. If F'_p were singular there would be two segments px_1 and px_2 in $N(p, K')$ with the same image under F'_p. Therefore it is enough to show that $F'_p(x_1) \neq F'_p(x_2)$ if $x_1 \neq x_2$, where x_1 and x_2 are arbitrarily near p. Let $A_\lambda B$ be the simplex of K (not K') containing x_λ as an inner point, where A_1B and A_2B do not meet except in B. If one of $A_\lambda B$ contains the other (i.e. if $A_1 = 1$ or $A_2 = 1$) let b be the mid-point of the segment x_1x_2, if not let b be the A_1-projection of x_1 in B. Since $\| x_\lambda - b \| \leqq c \| x_2 - x_1 \|$, and by lemmas 3 and 1, we may suppose x_1 and x_2 to be so near p that:

1. $x_\lambda b \subset N(p, K')$ ($\lambda = 1, 2$),
2. *$F_p(x_1b + bx_2)$ is an $| \infty, \alpha |$-approximation to $f(x_1b + bx_2)$,*
3. *$F'_p(x_1b + bx_2)$ is an $| \infty, \alpha |$-approximation to $f'(x_1b + bx_2)$.*

Since f' is an $| \infty, \alpha |$-approximation to f it follows that $F'_p(x_1b + bx_2)$ is an $| \infty, 3\alpha |$-approximation to $F_p(x_1b + bx_2)$. Since the angle between the segments $F_p(bx_1)$ and $F_p(bx_2)$ is at least 7α it follows from the special case of the theorem already proved that $F'_p(x_1) \neq F'_p(x_2)$. Therefore F'_p is $(1 - 1)$.

We now show that f' is locally $(1 - 1)$. By a familiar theorem and lemmas 3 and 1, there is a $\delta > 0$ such that:

1. $x_\lambda \in N(b, K)$ ($\lambda = 1, 2$), *for some* $b \in K$, if $\| x_2 - x_1 \| \leqq \delta$,
2. *F_b is an $| \infty, \alpha |$-approximation to f throughout the sub-set of K given by $\| x - b \| \leqq c\delta$ for any $b \in K$.*

This being so, I say that $f'(x_1) \neq f'(x_2)$ if $0 < \| x_2 - x_1 \| \leqq \delta$. For let $0 < \| x_2 - x_1 \| \leqq \delta$ and let A_1, A_2, B and b mean the same as before. Since $\| x_\lambda - b \| \leqq c \| x_2 - x_1 \| \leqq c\delta$, the map $f(x_1b + bx_2)$ is an $| \infty, \alpha |$-approximation to $F_b(x_1b + bx_2)$. Therefore $f'(x_1b + bx_2)$ is an $| \infty, 2\alpha |$-approximation to $F_b(x_1b + bx_2)$, and it follows from the special case of the theorem that $f'(x_1) \neq f'(x_2)$.

Finally we show that $f'(K')$ is $(1 - 1)$ throughout K' if it is an $| \epsilon, \alpha |$-approximation to $f(K)$, for a sufficiently small $\epsilon > 0$. The sub-set of the topological

[14] If f' is assumed to be recti-linear this step is unnecessary.

product $K^2 = K \times K$, for which $\| x_2 - x_1 \| \geqq \delta$ is compact, where δ means the same as in the preceding paragraph. Therefore the continuous function $\| f(x_2) - f(x_1) \|$ attains its minimum, say 3ϵ, on this sub-set, and $\epsilon > 0$ since f is $(1 - 1)$. Therefore, if $\| x_2 - x_1 \| \geqq \delta$ and if $f'(K')$ is an $| \epsilon, \alpha |$-approximation to $f(K)$, we have

$$\| f'(x_2) - f'(x_1) \| \geqq \| f(x_2) - f(x_1) \| - \| f(x_2) - f'(x_2) \| \\ - \| f'(x_1) - f(x_1) \| \geqq \epsilon > 0,$$

whence f' is non-singular and the proof is complete.

As a corollary to lemma 5 and theorems 1 and 3 we have:

THEOREM 4. *Given a non-singular C^1-complex $f(K) \subset R^n$, and $\epsilon, \rho > 0$, there is a subdivision K', of K, such that $L_f(K')$ is a non-singular, (ϵ, ρ)-approximation to $f(K)$.*

3. Let M^n be an n-dimensional manifold of class C^1. Without loss of generality we assume M^n to be smoothly imbedded[15] in R^m, and (ϵ, ρ)-approximations to maps in M^n will be measured in terms of the Euclidean metric for R^m. Let $f(K) \subset U \subset M^n$ be a non-singular C^1-complex, where U is the domain of an allowable coordinate system for M^n. Then theorem 4 is valid if the term linear is interpreted in terms of the coordinates for U, provided the sub-division K' is so fine that $L_f(K') \subset U$. For $f(K)$ is compact, and the metric taken from R^n by the coordinates is continuous in terms of R^m, and the parallelism taken from R^n is a first approximation to the parallelism of R^m. If $f(K) \subset M^n$ is a C^1-complex such that $f(A) \subset U(A)$ for each $A \subset cl(K - K_1)$, where $U(A)$ is the domain of an allowable coordinate system for M^n, then the proof of theorem 2 applies to approximations in M^n, taking ϵ to be so small that $f'(A') \subset U(A)$.

By a *C^1-triangulation*[16] of M^n, we shall mean a non-singular, locally finite[17] C^1-complex $f(K) = M^n$, which covers M^n. By an (n-dimensional) *unbounded, formal manifold* we shall mean a simplicial complex K, such that the complement of each vertex is combinatorially equivalent to the boundary of an n-simplex.

THEOREM 5. *If $f(K)$ is a C^1-triangulation of M^n, then K is an unbounded formal manifold.*

Let $f(K) = M^n$ be a C^1-triangulation. Then K, being a homeomorph of M^n, is n-dimensional and is a pseudomanifold[18] (i.e. each $(n - 1)$-simplex is on the boundary of precisely two n-simplexes). Therefore the complement, K_b, of any vertex, b, is a pseudo-manifold and hence a finite $(n - 1)$-cycle (mod 2). Now $F_b\{N(b, K)\} = F_b(bK_b) = f(b)F_b(K_b) \subset R^n$, where R^n is the tangent flat n-space to M^n at $f(b)$. Also F_b is an isomorphic map of bK_b, since f is non-singular.

[15] Hassler Whitney, Annals of Math., 3(1936), 645–80.

[16] It follows very easily from theorems 4 and 2, by Cairns' piecemeal construction (cf. lemma 7, below) that M^n has a C^1-triangulation. As we shall see this also follows from our theorem 6.

[17] We recall this is the only passage in §§2 and 3 in which K may be infinite.

[18] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), 125.

Therefore the radial projection of $F_b(K_b)$ from $f(b)$ in the boundary of an n-simplex $A^n \subset R^n$, of which $f(b)$ is an inner point, is a semi-linear, topological transformation, $\pi F_b(K_b) \subset \dot{A}^n$. Since K_b is a finite $(n-1)$-cycle it follows that $\pi F_b(K_b) = \dot{A}^n$, and the theorem is established.

Two maps, $f(K)$ and $f^*(K^*)$, will be described as *equivalent* if, and only if, K^* is the image of K in an isomorphism t, such that $f = f^*t$. This is obviously an equivalence relation in the technical sense (i.e. it is symmetric and transitive) and we shall now identify any two C^1-complexes which are given by equivalent C^1-maps. Thus $f_1(K_1) = f_2(K_2)$ will mean that the maps $f_1(K_1)$ and $f_2(K_2)$ are equivalent and the complexes $f_1(K_1)$ and $f_2(K_2)$ identical. If[19] $K = K_1 + \cdots + K_q$ we shall describe a non-singular C^1-complex $f(K)$ as the *non-singular union*, $f(K_1) + \cdots + f(K_q)$, of its sub-complexes $f(K_1), \cdots, f(K_q)$. Conversely, a set of non-singular C^1-complexes $f_1(K_1), \cdots, f_q(K_q)$ in M^n, will be said to have a non-singular union, $f_1(K_1) + \cdots + f_q(K_q)$, if, and only if, there is a non-singular C^1-complex $f^*(K^*) = f_1(K_1) + \cdots + f_q(K_q)$, such that $K^* = K_1^* + \cdots + K_q^*$ and each map $f^*(K_\lambda^*)$ is equivalent to $f_\lambda(K_\lambda)$ $(\lambda = 1, \cdots, q)$. Notice that, if $K = K_1 + \cdots + K_q$ and if a given map $f(K)$ is non-singular throughout each of $K_1, \cdots, K_q$, then $f(K_1), \cdots, f(K_q)$ may have a non-singular union even if $f(K)$ is singular. But in this case $f(K_1) + \cdots + f(K_q) \neq f(K)$. The following lemma is an obvious consequence of these definitions.

LEMMA 6. *If $f_1(K_1)$ and $f(K)$ are non-singular C^1-complexes with a non-singular union, and if $K = K_2 + K_3$, then $f_1(K_1)$ and $f(K_2)$ have a non-singular union and*

$$f_1(K_1) + f(K) = \{f_1(K_1) + f(K_2)\} + f(K_3).$$

Two complexes $f_1(K_1)$ and $f_2(K_2)$ in M^n will be said to *intersect in a common sub-complex* if, and only if, their intersection, as point sets, coincides with $f_1(K_{10}) = f_2(K_{20})$, where $K_{\alpha 0}$ is a sub-complex of K_α $(\alpha = 1, 2)$ and the map $f_1(K_{10})$ is equivalent to the map $f_2(K_{20})$. Let this be the case and, without altering the notation, let us star each simplex A_1, if there are any, belonging to $Cl(K_1 - K_{10})$ but not to K_{10} and such that $f_1(A_1)$ has the same vertices as a simplex $f_2(A_2) \subset f_2(K_2)$. Then, replacing K_α by an isomorphic complex,[20] if necessary, we may first separate K_1 from K_2 and then identify each simplex $A \subset K_{10}$ with $tA \subset K_{20}$, where $t(K_{10}) = K_{20}$ is an isomorphism such that $f_1 = f_2t$ in K_{10}. The result is a complex $K^* = K_1^* + K_2^*$, where K_α^* is the image of K_α in an isomorphism t_α such that $t_1 = t_2t$ in K_{10}. Therefore $f_1(K_1)$ and $f_2(K_2)$ are sub-complexes of $f^*(K^*)$, where $f^* = f_\alpha t_\alpha^{-1}$ in K_α^*. If each of the maps $f_\alpha(K_\alpha)$ is $(1-1)$ so is $f^*(K^*)$, since $f_1(K_1)$ and $f_2(K_2)$ do not meet except in $f_1(K_{10})$. Let $p_1 \in K_{10}$, $p_2 = tp_1$, $p^* = t_\alpha p_\alpha$ and let $N_\alpha = N(p_\alpha, K_\alpha)$, $N^* =$

[19] Here addition is used as in the theory of sets. K_λ and K_μ may have simplexes in common and may even coincide.

[20] e.g. a sub-complex of a k-simplex for an arbitrarily large k.

$N(p^*, K^*)$. Then, subject to the above conditions, it is clear that $f_1(K_1)$ and $f_2(K_2)$ have a non-singular union, namely $f^*(K^*)$, if, and only if,

1. *$f_1(K_1)$ and $f_2(K_2)$ are non-singular,*

2. *$f_1(N_1)$ and $f_2(N_2)$ have a non-singular union, namely $f^*(N^*)$, for each $p_1 \epsilon K_{10}$.*

In general the tangent star at $f^*(p^*)$ to $f^*(K^*)$ may be singular for some point $p^* \epsilon t_1K_{10}$ even if $f_1(K_1)$ and $f_2(K_2)$ are both non-singular.

Let $K = K_0 + E$ and let $f(K) \subset M^n$ be a C^1-complex such that $f(K_0)$ and $f(E)$ are non-singular. Also let $f(E) \subset U$, where U is the domain of an allowable coordinate system, which we regard as map, $x(D) = U$, of a region $D \subset R^n$ on U.

LEMMA 7. *Under these conditions, given ϵ, $\rho > 0$ there is an (ϵ, ρ)-approximation, $f'(K'') \subset M^n$, to $f(K)$, such that $f'(K_0'')$ and $f'(E'')$ are non-singular and have a non-singular union.*

Let $H \subset K$ be the sub-complex consisting of all the simplexes in K whose images in f meet $f(E)$, let $K_1 = H \cdot K_0$ and let $K_2 = Cl(K - H)$. Then $K = H + K_2$, $H = E + K_1$, and $K_0 = K_1 + K_2$. Without altering our notation we assume, after a suitable sub-division, that $f(H) \subset U$ and also $f(A) \subset U$, where $A \subset K$ is any simplex which meets H. By theorem 4, given ϵ_1, $\rho_1 > 0$, there is an (ϵ_1, ρ_1)-approximation, $f'(H') \subset U$, to $f(H)$, such that f' is "x-linear" throughout each simplex in H' (i.e. $f'(A) = x(B)$ for each $A \subset H'$, where $B \subset D$ is recti-linear). We partially extend f' by writing $f' = f$ throughout each simplex which does not meet H. Then, given ϵ_2, $\rho_2 > 0$ and assuming ϵ_1 and ρ_1 to be sufficiently small, it follows from theorem 2 that the approximation $f'(H')$ has an (ϵ_2, ρ_2)-extension, $f'(K')$, throughout K. Since $f(E)$ and $f(K_0)$ are non-singular, it follows from theorem 3 that $f'(E')$ and $f'(K_0')$ are non-singular if ϵ_2 and ρ_2 are sufficiently small, which we assume to be the case. We also take ϵ_2 to be so small that $f'(K_2')$ does not meet $f'(E')$. Finally we take $\epsilon_2 \leqq \epsilon$, $\rho_2 \leqq \rho$, in which case $f'(K'')$ is an (ϵ, ρ-approximation to $f(K)$, where K'' is any sub-division of K'.

The sub-set $f'(H') \subset M^n$, besides being the image of H' in f' is the homeomorph, $x(P)$, of a polyhedron $P = F + P_1 \subset D$, where

$$F = x^{-1}f'(E'), \qquad P_1 = x^{-1}f'(K_1'),$$

$x^{-1}f'$ being an isomorphism throughout each of E' and K_1' since $K_1' \subset K_0'$ and $f'(E')$ and $f'(K_0')$ are non-singular. Let $P' = F' + P_1'$ be a triangulation of P, F' and P_1' being recti-linear sub-divisions of F and P_1 which intersect in a common sub-complex. Since the map $x^{-1}f'\ (A) \subset P$ is non-degenerate for each simplex $A \subset H'$, the triangulation P' determines a sub-division $H'' = E'' + K_1''$, of H', such that F' and P_1' are isomorphic in $x^{-1}f'$ to E'' and K_1''. Therefore $x(F') = f'(E'')$ and $x(P_1') = f'(K_1'')$. Let $K'' = H'' + K_2'' = K_0'' + E''$ be an extension of the sub-division H'' throughout K'.

Since $x(P') = x(F') + x(P_1') = f'(E'') + f'(K_1'')$, and $f'(E'')\cdot f'(K_2'') = 0$, the (non-singular) C^1-complexes $x(P')$ and $f'(K)_2''$ intersect in a common sub-complex, namely $f'(K_1''\cdot K_2'')$. Also any simplex $A \subset P'$, such that $x(A)$ meets

$f'(K_2'')$, belongs to P_1' and not to F'. Therefore, if $p \epsilon K_1'' \cdot K_2''$ and $q = x^{-1}f'(p)$, then $N(q, P') = N(q, P_1')$. Since $x(P_1') = f'(K_1'')$ and $f'(K_0'') = f'(K_1'') + f'(K_2'')$ it follows that $x\{N(q, P')\}$ and $f'\{N(p, K_2'')\}$ have a non-singular union, namely $f'\{N(p, K_0'')\}$. Therefore $x(P')$ and $f'(K_2'')$ have a non-singular union. But

$$x(P') = f'(E'') + f'(K_1'')$$

and it follows from lemma 6 that

$$x(P') + f'(K_2'') = f'(E'') + \{f'(K_1'') + f'(K_2'')\} = f'(E'') + f'(K_0''),$$

and the lemma is established.

We now come to the main theorem.

THEOREM 6. *Given $\epsilon, \rho > 0$, and non-singular C^1-complexes $f_\lambda(K_\lambda) \subset M^n$ ($\lambda = 1, \cdots, q$), there are (ϵ, ρ)-approximations in M^n to $f_\lambda(K_\lambda)$, which have a non-singular union.*

If $q = 1$ the theorem is trivial and we shall prove it by induction on the total number of simplexes in $K_2, \cdots, K_q$, after an initial sub-division such that $f_\lambda(A)$ is in the domain of an allowable coordinate system, for each simplex $A \subset K_\lambda$ and each $\lambda = 2, \cdots, q$. Let $K_q = K_{q0} + A$, where A is a principal simplex in K_q and[21] $K_{q0} = Cl(K_q - A)$, and let U be the domain of an allowable coordinate system, which contains $f_q(A)$. By the hypothesis of the induction, given $\epsilon_1, \rho_1 > 0$ there are non-singular (ϵ_1, ρ_1)-approximations $f_\alpha'(K_\alpha'), f_q'(K_{q0}') \subset M^n$ to $f_\alpha(K_\alpha)$ and $f_q(K_{q0})$ ($\alpha = 1, \cdots, q - 1$), such that $f_1'(K_1'), \cdots, f_q'(K_{q0}')$ have a non-singular union. By theorem 2, given $\epsilon_2, \rho_2 > 0$ and provided ϵ_1 and ρ_1 are sufficiently small, there is an (ϵ_2, ρ_2)-extension, $f_q'(K_q')$, of $f_q'(K_{q0}')$, where $K_q' = K_{q0}' + A'$ and $f_q(A') \subset$ U. We take ϵ_2 and ρ_2 to be so small that $f_q'(K_q')$ is non-singular, according to theorem 3. We also take $\epsilon_1 \leqq \epsilon_2, \rho_1 \leqq \rho_2$, so that $f_\lambda'(K_\lambda')$ is an (ϵ_2, ρ_2)-approximation to $f_\lambda(K_\lambda)$ for each value of $\lambda = 1, \cdots, q$. Replacing $f'(K_\lambda')$ ($\lambda = 1, \cdots, q$) by equivalent maps, if necessary, and taking care that no internal simplex of A' coincides with a simplex of[22] K_α' ($\alpha = 1, \cdots, q - 1$) we may, without altering our notation, represent $f_q'(K_q')$ and the union $f_1'(K_1') + \cdots + f_q'(K_{q0}')$ as non-singular sub-complexes of a C^1-complex $g(K)$, where $K = K_1' + \cdots + K_q'$ and $g = f_\lambda'$ in K_λ'. Then $K = K_0 + A'$, where $K_0 = K_1' + \cdots + K_{q0}'$, and $g(K_0) = f_1'(K_1') + \cdots + f_q'(K_{q0}')$. Since $g(A') \subset U$ it follows from lemma 7 that, given $\epsilon_3, \rho_3 > 0$, there is an (ϵ_3, ρ_3)-approximation, $g'(K')$, to $g(K)$, such that $g'(K_0')$ and $g'(A'')$ have a non-singular union, where $K' = K_0' + A'' = K_1'' + \cdots + K_q''$. Then $g'(K_0')$, and hence $g'(K_\alpha'')$ ($\alpha = 1, \cdots, q - 1$), are non-singular, and since $g'(K_q'')$ is an (ϵ_3, ρ_3)-approximation to the non-singular complex $g(K_q')$ ($= f_q'(K_q')$) we may

[21] K_{q0} and $f_q(K_{q0})$ are empty if $K_q = A$.

[22] This may require an internal sub-division of A' if there are internal simplexes with all their vertices in the boundary. However, if the sub-division A' is given by the construction in the proof of theorem 2 there are no such simplexes.

take ϵ_3 and ρ_3 to be so small that $g'(K_q'')$ is also non-singular. This being so $g'(K_q'') = g'(K_{q0}'') + g'(A'')$, and by lemma 6 we have

$$g'(K_0') + g'(A'') = \{g'(K_1'') + \cdots + g'(K_{q0}'')\} + g'(A'')$$
$$= g'(K_1'') + \cdots + g'(K_q'').$$

Finally $g(K_\lambda')$ $(= f_\lambda'(K_\lambda'))$ is an (ϵ_2, ρ_2)-approximation to $f_\lambda(K_\lambda)$ and $g'(K_\lambda'')$ is an (ϵ_3, ρ_3)-approximation to $g'(K_\lambda')$. Therefore, taking $\epsilon_2 + \epsilon_3 \leqq \epsilon$, $\rho_2 + \rho_3 + \rho_2\rho_3 \leqq \rho$, the theorem is established.

Let V be any open sub-set of M^n and let $f_\lambda(K_{\lambda 0})$ be the sub-complex consisting of all the simplexes in $f_\lambda(K_\lambda)$ which meet $Cl(V)$. From the proof of lemma 7, and by adding to the hypotheses of the induction in theorem 6, we have the addendum:

Addendum: *If $f_1(K_{10}), \cdots, f_q(K_{q0})$ have a non-singular union the approximations in theorem* 6 *may be chosen so as not to disturb the part of this union which lies in V.*

For in the proof of lemma 7 it is only necessary to sub-divide K, or to alter the map f, in those simplexes which meet H. If $f(E) \subset M^n - V$ no simplex $f(A) \subset M^n$ which meets $f(E)$ is contained in V. If $f(E) \subset M^n - Cl(V)$ we may therefore assume, after an initial sub-division which leaves A unaltered if $f(A) \subset V$, that $f(H) \subset M^n - V$. Then $A \cdot H = 0$ if $f(A) \subset V$.

We now require M^n to be closed, a restriction which we remove later.

Theorem 7. *There is a C^1-triangulation of M^n.*

Since M^n is closed it can be covered by the interiors of a finite set of non-singular, n-dimensional C^1-simplexes $f_1(A_1), \cdots, f_q(A_q)$. By theorem 6, given $\epsilon > 0$, there are (ϵ, ∞)-approximations, $f_\lambda'(A_\lambda')$ $(\lambda = 1, \cdots, q)$, to $f_\lambda(A_\lambda)$, which have a non-singular union $f(K) = f_1'(A_1') + \cdots + f_q'(A_q')$. It follows from well known theorems[23] that, provided ϵ is sufficiently small, each point of M^n is internal to at least one of the cells $f_\lambda'(A_\lambda')$. Therefore $f(K)$ covers M^n and is a C^1-triangulation.

Theorem 8. *If $f_1(K_1)$ and $f_2(K_2)$ are two C^1-triangulations of M^n, then K_1 and K_2 are combinatorially equivalent.*

For, by theorem 6, there are non-singular approximations $f_\lambda'(K_\lambda') \subset M^n$ to $f_\lambda(K_\lambda)$ $(\lambda = 1, 2)$, which have a non-singular union. Since K_λ is a pseudo-manifold it is a cycle (mod 2). Therefore K_λ' is a cycle (mod 2), and since the map f_λ' is topological it follows that M^n is completely covered by $f_\lambda'(K_\lambda')$. Therefore $f_1'(K_1') = f_2'(K_2')$, since $f_1'(K_1')$ and $f_2'(K_2')$ intersect in a common sub-complex, and K_1' is isomorphic to K_2'.

With suitable restrictions, similar theorems to theorems 7 and 8 may be proved for a bounded manifold $M_0^n \subset M^n$. For example, let M^n and also the frontier, M^{n-1}, of M_0^n be manifolds of class C^3, and let M^n be given a Riemannian metric $ds^2 = g_{ij}\,dx^i\,dx^j$, where the functions g_{ij} are of class C^2 in allowable coordinate systems for M^n. Then, for some $\delta > 0$, no two of the geodesic seg-

[23] See, for example, Alexandroff and Hopf (loc. cit.), pp. 100 (theorem IV) and 459 (Rouché's theorem).

ments pq, of length δ, will meet each other, where $pq \subset M_0^n$ is normal at $p \in M^{n-1}$ to M^{n-1}. If $f(K) \subset M^{n-1}$ is a C^1-triangulation of M^{n-1} the sub-set of M^n covered by the segments pq is a non-singular C^1-image of the polyhedral complex $K \times \langle 0, 1 \rangle$, which may be triangulated by a normal sub-division. It is now easy to show that, without disturbing $f(K)$, some approximation to this triangulation may be extended throughout M_0^n.

Assuming only that M^n and M^{n-1} are of class C^1, let $f_1(K_1)$ and $f_2(K_2)$ be two C^1-triangulations of M_0^n. By theorem 6, applied to the sub-complexes $f_1(K_{10})$, $f_2(K_{20})$ covering M^{n-1}, by theorem 2, and since $f_\lambda(K_{\lambda 0})$ is the point-set frontier of $f_\lambda(K_\lambda)$, we may assume that $f_1(K_{10}) = f_2(K_{20})$. By adding to the hypotheses of the induction in theorem 6 we see that, if the maps $f_\lambda(K_\lambda)$ ($\lambda = 1, \cdots, q$) are equivalent to each other throughout mutually isomorphic sub-complexes $K_{\lambda 0} \subset K_\lambda$, then the approximations $f'_\lambda(K'_\lambda) \subset M^n$, which have a non-singular union, may be chosen so that $f'_1(K'_{10}) = \cdots = f'_q(K'_{q0})$. In the case of the triangulations $f_\lambda(K_\lambda) = M_0^n$ ($\lambda = 1, 2$), with $K_{\lambda 0} = \dot{K}_\lambda$ (mod 2), it follows that $f'_1(K'_1) = f'_2(K'_2)$, whence K'_1 is isomorphic to K'_2.

4. We conclude by showing how many of these results can be extended to infinite complexes and open manifolds. An infinite complex $f(K) \subset M^n$ is to be such that only a finite number of simplexes $f(A)$ ($A \subset K$) meet any compact sub-set of M^n. A manifold $M^n \subset R^m$ is to be a closed, but not necessarily compact, sub-set of R^m. An (ϵ, ρ)-approximation to $f(K)$ shall mean the same as before, except that ϵ and ρ may now be any non-negative functions, $\epsilon(p)$ and $\rho(p)$, which are defined for each $p \in K$, and $\epsilon, \rho > 0$ is to mean that $\epsilon(p)$ and $\rho(p)$ have positive lower limits in each compact sub-set of K. It is often convenient to define such a function in terms of a particular covering of K by compact sub-sets $[F]$ (e.g. the simplexes or stellar neighbourhoods), only a finite number of which meet any one compact sub-set, and a positive function of sets, $\bar{\eta}(F)$, defined for each set in the covering. Then $\eta(p)$ may be defined as the minimum of $\bar{\eta}(F)$ for F containing p. Conversely, given $\eta(p)$, the function $\bar{\eta}(F)$ may be defined as the lower limit of $\eta(p)$ for $p \in F$. For example, in the proof of theorem 2, with K finite or infinite, we may take $\epsilon(p)$, $\rho(p)$ to be defined in terms of given functions $\bar{\epsilon}(A)$, $\bar{\rho}(A)$, where A is any simplex in K, and $\epsilon_1(p)$, $\rho_1(p)$ to be not greater than suitably chosen functions $\bar{\epsilon}_1(\dot{A})$, $\bar{\rho}_1(\dot{A})$ if $p \in \dot{A}$. The theorem then follows by induction on the dimensionality of $K - K_1$ and the same construction as in the finite case. In proving theorem 3 we may define α in terms of a function $\bar{\alpha}(N) > 0$, where N is the stellar neighbourhood of any vertex in K. If $\bar{\alpha}(N)$ is suitably chosen, any $| \infty, \alpha |$-approximation to a given non-singular C^1-map is locally non-singular and it is easily shown, as in the finite case, that an $| \epsilon, \alpha |$-approximation is non-singular for a suitable $\epsilon(p)$.

In the absence of lemma 5, which seems to be comparatively difficult if K is infinite, we replace theorem 4 by the less explicit theorem:

THEOREM 9. *Given $\epsilon, \rho > 0$, there is a non-singular, recti-linear (ϵ, ρ)-approximation to any non-singular C^1-complex $f(K) \subset R^n$.*

This may be proved in the same way as the extension of theorem 6 to infinite sets of (possibly infinite) complexes $f_\lambda(K_\lambda) \subset M^n$ $(\lambda = 1, 2, \cdots)$, only a finite number of which meet any one compact sub-set of M^n. To prove this let M^n be a closed set in R^m, referred to Cartesian coordinates $y^1, \cdots, y^m$. Let V_r be the sub-set of M^n for which $\| y \| < r$ and, after a suitable sub-division, let each simplex of $f_\lambda(K_\lambda)$ which meets $Cl(V_\mu)$ lie in $V_{\mu+1}$, for each $\lambda, \mu = 1, 2, \cdots$. By theorems 6 and 2 we may assume, after a suitable (ϵ_r, ρ_r)-approximation to $f_\lambda(K_\lambda)$ $(\lambda = 1, 2, \cdots; \epsilon_r(p) < \epsilon(p), \rho_r(p) < \rho(p))$, that the maximal sub-complexes of $f_1(K_1), f_2(K_2), \cdots$ whose simplexes all meet $Cl(V_r)$ have a non-singular union, for some[24] $r = 1, 2, \cdots$. If follows from theorem 6 and its addendum that, by a suitable (ϵ_r', ρ_r')-approximation to the first (ϵ_r, ρ_r)-approximation, this condition can be maintained with r replaced by $r + 1$, without disturbing the part of the union, say $g(P_r)$, which lies in V_r. The result will be an $(\epsilon_{r+1}, \rho_{r+1})$-approximation to $f_\lambda(K_\lambda)$, for each $\lambda = 1, 2, \cdots$, where $\epsilon_{r+1} = \epsilon_r + \epsilon_r'$, $\rho_{r+1} = \rho_r + \rho_r' + \rho_r\rho_r'$. Since $\epsilon_r < \epsilon$, $\rho_r < \rho$, we may choose ϵ_r', ρ_r' so that $\epsilon_{r+1} < \epsilon$, $\rho_{r+1} < \rho$ and the induction is complete. In the succeeding stages of the construction we may take $P_r \subset P_{r+1} \subset \cdots$ and the required union in $g(P)$, where $P = P_1 + P_2 \cdots$. This theorem carries with it theorems 7 and 8 for open manifolds.[25]

Finally, if M^n is a manifold of class C^k $(k = 2, \cdots, \infty$ or $\omega)$, a C^k-complex $f(K) \subset M^n$ may be defined in the same way as a C^1-complex, and we have:

THEOREM 10. *Given $\epsilon, \rho > 0$ and a non-degenerate C^1-complex $f(K) \subset M^n$, there is an (ϵ, ρ)-approximation to $f(K)$ which is of class C^k.*

If M^n be imbedded as a class C^k manifold in R^m, the flat $(m - n)$-spaces normal to M^n form a system of class C^{k-1}. It is, however, possible[26] to define a class C^k system of flat $(m - n)$-spaces approximately normal to M^n. By means of such spaces, we can project back into M^n a recti-linear (ϵ', ρ')-approximation to $f(K)$, thus obtaining a proof of Theorem 10, provided (ϵ', ρ') are chosen sufficiently small.

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[24] Notice that we only need theorem 6 for finite complexes and theorem 2 for finite $K - K_1$. We may take $\epsilon_r(p)$, $\rho_r(p)$ to be any constants less than the lower limits of $\epsilon(p)$, $\rho(p)$ for $p \in K_\lambda \cdot Cl(V_{r+1})$, and $\epsilon_r(p) = \rho_r(p) = 0$ for $p \in K_\lambda - K_\lambda \cdot Cl(V_{r+1})$.

[25] Here again Cairns' method leads to a more direct proof of the triangulation theorem.

[26] Hassler Whitney, loc. cit., §25.

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ON THE HOMOTOPY TYPE OF MANIFOLDS

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1. The object of this note is to call attention to certain theorems, which follow very easily from some results due to E. Stiefel,[1] H. Seifert,[2] Hassler Whitney,[3] and myself.[4] They refer to a class of manifolds which we call the class II, and are intended to throw light on the question, raised by W. Hurewicz,[5] whether two closed manifolds of the same homotopy type are necessarily homeomorphic. The theorems depend both on M. H. A. Newman's[6] theory of combinatorial equivalence, as re-developed by J. W. Alexander[7] and carried further in S. S., and on theorems concerning differentiable manifolds. Therefore it is necessary to give a precise meaning to the term 'manifold'.

By an n-dimensional manifold, M^n, we shall mean a class of combinatorially equivalent, simplicial complexes covering the same space, each complex being a *formal manifold*, meaning that the complement[7] of each vertex is combinatorially equivalent to $\dot{A}^n$ or to A^{n-1}, according as the vertex in question is inside M^n or on $\dot{M}^n$, where A^k stands for a closed k-simplex and $\dot{M}^n$ is the boundary of M^n. These covering complexes will be called *proper triangulations* of M^n (of course any simplicial complex covering M^n is a proper triangulation if the '*Hauptvermutung*' is true). The proper triangulations of an unbounded manifold of class C^1, or *smooth manifold*, are to be C^1-triangulations.[7a] By a smooth, bounded, n-dimensional manifold we shall mean the manifold of which a sub-complex $K_0^n \subset K^n$ is a proper triangulation, where K^n is a C^1 triangulation of a smooth, unbounded n-dimensional manifold and K_0^n is a formal manifold. By the topological product $M^n \times A^k$ we shall mean the manifold having a normal subdivision of the cell-complex $K^n \times A^k$ as a proper triangulation, where K^n is a proper triangulation of M^n. We shall use $\equiv$ to indicate combinatorial equivalence, and $M_1^n \equiv M_2^n$ will mean that $K_1^n \equiv K_2^n$, where K_i^n is a proper triangulation of M_i^n.

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[1] E. Stiefel, Comm. Math. Helvetici, 8 (1935), 305–53.

[2] H. Seifert, Math. Zeit., 41 (1936), 1–17.

[3] Hassler Whitney, Proc. N. A. S., 21 (1935), 464–8; Bull. American Math. Soc., 43 (1937), 785–805. Page references will refer to the second of these papers.

[4] J. H. C. Whitehead, Proc. London Math. Soc., 45 (1939), 243. This paper will be referred to as S. S.

[5] W. Hurewicz, Akad. Wet. Amsterdam, 29 (1936), 125.

[6] M. H. A. Newman, Akad. Wet. Amsterdam, 29 (1926), 611–41, 30 (1927), 670–3.

[7] J. W. Alexander, Annals of Math., 31 (1930), 292–320.

[7a] J. H. C. Whitehead, Annals of Math. this number, 809–824. This paper will be referred to as C. C. Relevant to the present paper are theorems 4, 5, 7 and 8 of C. C.

We now state some of our theorems, postponing the proof of theorem 1 and the definition of the class Π till §2. It is to be understood that the manifolds referred to in these theorems are connected and covered by finite complexes.

THEOREM 1. *If* $M_i^n \in \Pi$ $(i = 1, 2)$ *and* M_1^n *and* M_2^n *have the* same nucleus,[4] *then*

$$M_1^n \times A^k \equiv M_2^n \times A^k$$

for sufficiently large values of k.

It is shown in S. S. that, provided their fundamental group satisfies a certain condition,[8] two (finite) complexes have the same nucleus if they are of the same homotopy type. For manifolds with such a group, theorem 1 can therefore be restated with 'have the same nucleus' replaced by 'are of the same homotopy type'.

A bounded manifold M^n, which is an absolute retract (i.e. is of the same homotopy type as a single point) belongs to the class Π if it is combinatorially equivalent to a smooth manifold. If M^n is smooth we may assume that[9] $M^n \subset M_1^n \subset R^{2n+k}$ for any $k > 0$, where M_1^n is an unbounded analytic manifold and R^m is Euclidean m-space. Since M^n has the same homology and cohomology groups as a cell its normal sphere-space[9a] in R^{2n+k} is simple. Taking $k = 5$ we have, from theorem 5, below, and S. S., theorem 25, corollary 3:

THEOREM 2. *If*[10] $\pi_1(M^n) = 1, \beta(M^n) = 0$ $(r = 1, \cdots, n)$ *and* M^n *is smooth, then*

$$M^n \times A^{n+5} \equiv A^{2n+5}.$$

It will be seen that any (bounded) polyhedral $M^n \subset R^n$ belongs to Π. Therefore $M^n \times A^{n+5} \equiv A^{2n+5}$ if M^n is the finite region bounded by a polyhedral $(n - 1)$-sphere in R^n, or even if $\dot{M}^n$ is of the same homotopy type as $\dot{A}^n$.

The Poincaré hypothesis, in its combinatorial form and as generalized by Hurewicz[5] from $n = 3$ to any n, is equivalent to the hypothesis.

If $\dot{M}^n$ *is an* $(n - 1)$*-sphere and if* M^n *is an absolute retract, then* $M^n \equiv A^n$.

Discarding the condition that $\dot{M}^n$ is an $(n - 1)$-sphere, we have what may be called the extended Poincaré hypothesis, namely:

A bounded, n-dimensional manifold, which is an absolute retract is an n-element.

From theorem 2, since a k-element is the topological product of k linear segments, we have:

THEOREM 3. *The extended Poincaré hypothesis, for smooth manifolds at least, is equivalent to the hypothesis*:

If $M^p \times A^1 \equiv A^{p+1}$, *then* $M^p \equiv A^p$.

This theorem raises various questions, one of which can be answered very

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[8] See S. S. p. 287. See also a paper by G. Higman to be published shortly by the London Math. Soc.

[9] Hassler Whitney, Annals of Math., 37 (1936), 645–80.

[9a] Appendix, Theorem 2, corollary.

[10] $\pi_1(M^n)$ denotes the (multiplicative) fundamental group and $\beta_r(M^n)$ the (additive) r^{th} homology group of M^n.

simply, namely: 'are there manifolds $M_1^n \not\equiv M_2^n$ such that $M_1^n \times A^1 \equiv M_2^n \times A^1$?' The answer is in the affirmative. For let $M_i^3 = M_i^2 \times A^1$, where M_1^2 is a torus with one hole and M_2^2 is a 2-sphere with three holes. Then $M_1^2 \not\equiv M_2^2$. On the other hand, taking $M_i^2 \subset R^3$, it is easily verified that $M_1^3 \equiv M_2^3$, since $M_i^3 \subset R^3$ is obviously a regular neighborhood (S. S., p. 293) of two simple circuits with a single point in common. As another, and perhaps more interesting example, let M_i^3 $(i = 1, 2)$ be a lens space of type[11] (p, q_i), from which the interior of a 3-simplex A_i^3 has been removed, where $q_1q_2 \not\equiv \pm 1$ mod p. Then M_i^3 contracts (S. S., pp. 248 and 258) into the 2-cell, bounded by a circuit taken p times, which, taken twice, bounds a lens model of M_i^3. Therefore M_1^3 and M_2^3 have the same nucleus. It will be seen that $M_i^3 \in \Pi$, whence, by theorem 1, $M_1^3 \times A^k \equiv M_2^3 \times A^k$ for large values of k (actually for $k \geqq 6$). But M_1^3 and M_2^3 are not combinatorially equivalent. For if they were, the lens spaces $M_1^3 + A_1^3$ and $M_2^3 + A_2^3$ would be combinatorially equivalent, which they are not since[12] $q_1q_2 \not\equiv \pm 1$ mod p.

2. Let a proper triangulation, K^n, of a given manifold, M^n, be represented as a recti-linear complex in R^{n+k}, and let $U(K^n, R^{n+k})$ be a regular neighborhood[12a] of K^n. Then our definition of Π is: $M^n \in \Pi$ *if, and only if,*

$$U(K^n, R^{n+k}) \equiv K^n \times A^k \tag{2.1}$$

for large values of k. Provided $k \geqq n + 3$ it follows from S. S., theorems 23 and 24, that this definition is independent of the choice of the proper triangulation K^n, of the choice of the regular neighborhood $U(K^n, R^{n+k})$ and of the way in which K^n is imbedded in R^{n+k}. If $K^n \subset R^{n+k} \subset R^{n+k+l} = R^{n+k} \times R^l$ $(l > 0)$ we may take

$$U(K^n, R^{n+k+l}) = U(K^n, R^{n+k}) \times A_1^1 \times \cdots \times A_l^1.$$

For the latter is a manifold and, by an obvious induction on l, it contracts into $U(K^n, R^{n+k})$, and hence into K^n. Therefore, if the condition (2.1) is satisfied by some $K^n \subset R^{n+k}$, it is satisfied for every $k_1 > k$ and a suitable $K^n \subset R^{n+k_1}$. Theorem 1, above, is now seen to be an immediate consequence of S. S., theorem 25.

It follows from an argument in S. S. (p. 298) that an n-sphere belongs to Π for each value of n. Moreover, if $M^n \in \Pi$ and $M_0^n \subset M^n$, then $M_0^n \in \Pi$. For let t be a semi-linear homeomorphism of $K^n \times A^k$ on $U(K^n, R^{n+k})$, where K^n is a proper triangulation of M^n which contains a sub-complex, K_0^n, covering M_0^n. Then $t(K_0^n \times A^k)$ is a manifold and contracts geometrically into

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[11] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), 210.

[12] K. Reidemeister, Abh. Math. Sem. Hamb., 11 (1935), 102–9; Journal f. d. r. u. a. Math., 173 (1935), 164–73.

[12a] S. S. p. 293. Observe that regular neighbourhoods are not necessarily neighbourhoods in the sense of topology.

$t(K_0^n \times p)$, for any point $p \in A^k$, and (2.1) is satisfied by $t(K_0^n \times p) \subset R^{n+k}$. Therefore $t(K_0^n \times A^k)$ is a regular neighborhood of K_0^n. More generally, let $M_0^r \subset M^n$ and let a proper triangulation K_0^r, of M_0^r, be a sub-complex of K^n. If a regular neighborhood $U_0^n = U(K_0^r, K^n) \equiv K_0^r \times A^{n-r}$ we shall say that M_0^r is in *regular position*[13] in M^n. This is always the case if $r = n$, for then we may take U_0^n to be K_0^r itself.

THEOREM 4. *If $M_0^r \subset M^n$ is in regular position in M^n and $M^n \in \Pi$, then $M_0^r \in \Pi$.*

For, with the above notation, U_0^n is an n-dimensional manifold in M^n and we have shown that if $M^n \in \Pi$, then $U_0^n \in \Pi$. That is to say

$$U^{n+k} = U(U_0^n, R^{n+k}) \equiv U_0^n \times A^k$$

for some value of k and some recti-linear $U_0^n \subset K^n \subset R^{n+k}$. But U^{n+k} contracts into U_0^n and the latter contracts into K_0^r. Therefore U^{n+k} is also a regular neighborhood of K_0^r, and if $U_0^n \equiv K_0^r \times A^{n-r}$ we have

$$U^{n+k} \equiv U_0^n \times A^k \equiv K_0^r \times A^{n-r} \times A^k \equiv K_0^r \times A^{n-r+k},$$

and the theorem is established.

With the help of theorem 4 we can dispose of the case $n = 2$. No non-orientable manifold can belong to Π. For its regular neighborhood in R^{n+k}, being an $(n + k)$-dimensional manifold in R^{n+k}, is orientable, while its topological product with a cell is not. On the other hand any orientable surface may be represented as a polyhedron in R^3 and is necessarily in regular position. Therefore it belongs to Π. Also any orientable, polyhedral surface in R^m is in regular position if[14] $m \geqq 7$. Of course theorem 1 is trivial for any closed surface, whether orientable or not. Also it follows from special arguments, as in the remarks following theorem 3, that theorem 1, with $k = 1$, is true of bounded, orientable surfaces.

Now let $M^n \subset R^{n+k}$ be a smooth manifold which, without loss of generality, we may assume to be analytic.[9]

THEOREM 5. *$M^n \in \Pi$ if its normal sphere-space*[3] *in R^{n+k} is simple.*

Since M^n is compact there is a positive δ such that the flat k-spaces normal to M^n at two different points do not meet at a distance less than 2δ from M^n. Therefore no two of the k-cells $E^k(p)$ meet each other, where $E^k(p)$ is the interior and boundary of a $(k - 1)$-sphere with centre p and radius δ in the normal flat k-space at p. To say that the normal sphere-space is simple is to say that k mutually orthogonal, unit vectors $e_1(p), \cdots, e_k(p)$ are defined in the normal flat k-space at each point $p \in M^n$, and that $e_\lambda(p)$ varies continuously with p.

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[13] Hassler Whitney, Annals of Math., 37 (1936), 865–78.

[14] Though this lower limit for m can probably be reduced from 7 to 5 it cannot be discarded. For if K is a knotted circuit in a 3-sphere, S^3, it may be verified that the 2-sphere $(a + b)K$ is not in regular position in the 4-sphere $(a + b)S^3$, where a and b are vertices not in S^3. (Cf. E. Artin, Abh. Math. Sem. Hamb., 4 (1925), 174–7.)

After a process of approximation, projection in $E^k(p)$, and a final normalization, we may assume that $e_\lambda(p)$ varies analytically with p. The bounded manifold M^{n+k}, which is swept out by $E^k(p)$ as p describes M^n, is then seen to be the image of $M^n \times E^k(p_0) \equiv M^n \times A^k$ in an analytic transformation which maps M^n on itself. Therefore a suitable triangulation of $M^n \times A^k$ determines a C^1-triangulation, P^{n+k} of M^{n+k}, which contains a proper triangulation of M^n as a sub-complex. Let K^{n+k} be a rectilinear model of P^{n+k} and let $K^n \subset K^{n+k}$ be the subcomplex representing M^n. By C. C., theorem 4, there is a semi-linear, topological map $F(K^{n+k}) \subset R^{n+k}$. Then $F(K^{n+k})$ $(\equiv M^n \times A^k)$ is a regular neighborhood of $F(K^n)$ $(\equiv M^n)$, and the theorem is established.

It follows from this theorem, and the results referred to at the beginning of §1, that $M^n \in \Pi$, where M^n is a smooth, orientable manifold, if any one of the following conditions is satisfied:

1. *M^n is closed and admits an internal parallelism, as is always the case if*[1] *$n = 3$, or for example, if M^n is a Lie group.*
2. *M^n is closed and can be represented as a manifold of class C^2 in R^{n+1} or in* R^{n+2} (Seifert[2]).
3. M^n is bounded and all its cohomology groups vanish with integral, and hence with all coefficients. It can be shown that this follows from the general theory of sphere-spaces.[3]

The sufficiency of the first condition follows from a theorem similar to theorem 23 on pp. 43 and 44 of Stiefel's paper.[1] For let $M^n \subset R^{n+k}$, where $k \geqq n + 1$, and let K^n be a triangulation of M^n. Then we successively set up outer parallelisms (i.e. parallelisms in the normal flat k-spaces) over $K^0, K^1, \cdots, K^n$, where K^r is the r-dimensional skeleton of K^n. An outer parallelism over K^r $(0 \leqq r < n)$ determines an $(r + 1)$-dimensional cocycle in K^{r+1}, whose coefficients are elements of $\pi_r(G_k)$, where G_k is the group of rotations in R^k. The parallelism over K^r may be extended throughout K^{r+1} if this cocycle is zero. If it is not zero, but cohomologous to zero, then the parallelism over K^r may be replaced by one for which the corresponding cocycle is zero.[14a] Thus K^{r+1} admits an outer parallelism if the cocycle determined by the outer parallelism over K^r is cohomologous to zero. Since $r + 1 < k$ it follows from the analysis of G_k $(= V_{k,k-1})$ in §1 of Stiefel's paper, that a map $f(S^r) \subset G_k \subset G_{n+k}$, which is homotopic to a point in G_{n+k}, is homotopic to a point in G_k; also that any $f(S^r) \subset G_{n+k}$ can be deformed into a map in G_k. Therefore a lemma, analogous to the one in Stiefel's theorem 23, follows from arguments similar to those in his §3. Therefore the $(r + 1)$-dimensional cocycle in K^{r+1}, which is determined by an outer parallelism over K^r is cohomologous to zero. Finally, Stiefel's assumption that some triangulation of M^n is a sub-complex of a triangulation of R^{n+k} need not, in this case, be taken as an additional axiom. For we may assume M^n to be analytic and sub-divide it and a recti-linear triangulation of

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[14a] Cf. S. Eilenberg, Annals of Math., 41 (1940), 231–51.

R^{n+k} by the van der Waerden-Lefschetz method.[15] The result will not, in general, be a C^1-triangulation, but it will suffice in setting up the outer parallelism. Alternatively we may replace M^n by a homeomorphic polyhedral $F(K^n)$, as in the proof of theorem 5, and attach a flat n-space and a flat k-space to each point of $F(K^n)$, which are respectively parallel to the tangent and normal flat spaces at the corresponding point of M^n. Then an inner parallelism in M^n determines a parallelism in the n-spaces attached to the points of $F(K^n)$, and a parallelism in the k-spaces at points of $F(K^n)$ will determine an outer parallelism for M^n.

APPENDIX

(*Extract from a letter of the author to Hassler Whitney under date of Jan. 26, 1940.*— The Editors.)

* * * I omitted to prove that G_m, the group of rotations in Euclidean metric space R^{m+1}, is r-simple for each $r \geqq 1$, as the term is used by S. Eilenberg.[16] This condition may be expressed as follows. Let X be any arcwise connected topological space, let $\tilde{X}$ be its universal covering space and let Γ_1 be the group of covering transformations of $\tilde{X}$ (i.e. the group of homeomorphisms $\gamma_1(\tilde{X}) = \tilde{X}$, such that $u\gamma_1 = u$, where $u(\tilde{X}) = X$ is a locally (1-1) map of $\tilde{X}$ on X). Then X is said to be 1-simple if $\pi_1(X)$ is Abelian, and r-simple ($r > 1$) if, and only if, any spherical map $f(S^r) \subset \tilde{X}$ is homotopic in $\tilde{X}$ to the map $\gamma_1 f(S^r)$, for each $\gamma_1 \epsilon \Gamma_1$. Let us assume that Γ_1 is a sub-group of some arcwise connected, topological group Γ, of homeomorphisms $\gamma(\tilde{X}) = \tilde{X}$, whose topology agrees with that of $\tilde{X}$, meaning that $\gamma(x)$ varies continuously with $x \epsilon \tilde{X}$ and $\gamma \epsilon \Gamma$. Then the identity in Γ, say γ_0 is joined to a given $\gamma_1 \epsilon \Gamma_1$ by a segment $\gamma_t \epsilon \Gamma$ $(0 \leqq t < 1)$. Therefore $\gamma_1 f_0(S^r) = f_1(S^r)$, say, is the image of a given map, $f_0(S^r) \subset \tilde{X}$, in the deformation $f_t = \gamma_t f_0$, whence X is r-simple for any $r > 1$. Therefore, and since Γ_1 is isomorphic to $\pi_1(X)$, we have the theorem:

Theorem 1. *If Γ_1 satisfies the above condition and is also Abelian, then X is r-simple for each $r \geqq 1$.*

Let X be an arcwise connected topological group and let $\tilde{X}$ be its universal covering group. Then Γ_1 is Abelian, and is also a sub-group of the 'left translations' $\xi \to \gamma\xi$ (also of the 'right translations' $\xi \to \xi\gamma$, since Γ_1 is not only Abelian but, if the translation $\xi \to \gamma\xi$ is identified with the element $\gamma \epsilon \tilde{X}$, then Γ_1 belongs to the centre of $\tilde{X}$). Since X is arcwise connected, so is $\tilde{X} = \Gamma$, and we have the corollary:

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[15] B. L. van der Waerden, Math. Ann., 102 (1929), 337-62. S. Lefschetz, *Topology*, New York (1930), 364. See also B. O. Koopman and A. B. Brown, Trans. American Math. Soc., 34 (1932), 231-52 and S. Lefschetz and J. H. C. Whitehead, *ibid.*, 35 (1933), 510-17.

[16] S. Eilenberg. Fund. Math. 32 (1939), 167-75.

COROLLARY. *Any arcwise connected topological group is r-simple for each $r \geqq 1$.*

The consequence of this condition which interests us here is that, if X is r-simple, then a unique element of $\Pi_r(X)$ is determined by a 'free' map $f(S^r) \subset X$, meaning a map which is independent of the base point for $\Pi_r(X)$.

Now let an orientable sphere-space $S(K^n)$ be given, where K^n is a simplicial complex and the associated spheres are ν-dimensional, and let $S(K^n)$ be simple in the r-dimensional skeleton, K^r, of K^n $(0 < r < n)$. We shall assume that $S(K^n)$ is not only orientable but oriented, meaning that the associated spheres $S^\nu(p)$ $(p \in K^n)$ and the base sphere S_0^ν are oriented, and that the defining maps $\xi\{p, S^\nu(p)\} = S_0^\nu$ are all direct. Thus the (orthogonal) transformations of S_0^ν into itself by which 'transformations of coördinates' are determined will be rotations. Let A_i^{r+1} $(i = 1, 2, \cdots)$ be the (oriented) $(r + 1)$-simplexes in K^{r+1} and, using the rotation, let $q \to \xi_i(p, q) \in S_0^\nu$ $(p \in A_i^{r+1}, q \in S^\nu(p))$ be a local coördinate system for A_i^{r+1}. Since $S(K^r)$ is simple there is a map $q \to \eta(p, q) \in S_0^\nu$ defined for each $p \in K^r$, $q \in S^\nu(p)$, such that the rotation

$$q_0 \to \phi_p(q_0) = \xi_i\{p, \eta^{-1}(p, q_0)\} \qquad (p \in \dot{A}_i^{r+1}, q_0 \in S_0^\nu)$$

varies continuously with p. In other words, $p \to \phi_p$ is a continuous map of $\dot{A}_i^{r+1}$ in $G_{\nu+1}$, and since $G_{\nu+1}$ is r-simple $p \to \phi_p$ defines a unique element $\alpha_i \in \pi_r(G_{\nu+1})$. The element α_i is independent of the coördinate system ξ_i. For if $\xi_i'(p, q)$ is a second coördinate system for A_i^{r+1}, then $p \to \xi_i'\xi_i^{-1} = \psi_0(p)$, say, is a map of A_i^{r+1} in $G_{\nu+1}$. Since A_i^{r+1} can be shrunk into a point there is a deformation ψ_t $(0 \leqq t \leqq 1)$, of $\psi_0(p)$ into the map given by $\psi_1(p) = 1$, the identity in $G_{\nu+1}$. Therefore the coördinate system ξ_i' $(= \xi_{i0}')$ may be deformed into ξ $(= \xi_{i1}')$: Thus

$$\xi_{it}'(p, q) = \psi_t(p)\{\xi_i(p, q)\},$$

remembering that $\psi_t(p)$ is a rotation of S_0^ν into itself. Therefore the map of $\dot{A}_i^{r+1}$ in $G_{\nu+1}$, which is defined by ξ_i' and η, is homotopic to the above map $p \to \phi_p$, and hence determines the same element $\alpha_i \in \pi_r(G_{\nu+1})$. Let B_λ^r be an oriented r-simplex, which is common to A_i^{r+1} and to A_j^{r+1} and let $\eta_\lambda'(p, q)$ be any coördinate system for B_λ^r, which coincides with $\eta(p, q)$ in $\dot{B}_\lambda^r$. Then the map $\xi_i\eta^{-1}$, of B_λ^r in $G_{\nu+1}$, together with the map $\xi_i\eta_\lambda'^{-1}$, in which the orientation of B_λ^r is reversed, determine an element $\beta_\lambda \in \pi_r(G_{\nu+1})$. It follows from a similar argument to the one just given that the same element, β_λ, is determined by η, η_λ' and a coördinate system ξ_j for A_j^{r+1}. Thus, if η_λ' is constructed in such a way that η, η_λ' and ξ_i determine a given element β_λ, then η, η_λ' and ξ_j lead back to the same element β_λ.

After these preliminaries it follows from arguments which are similar to some of those used by E. Stiefel[1] and by Eilenberg[14a] that

1. $$C^{r+1} = \sum_i \alpha_i A_i^{r+1}$$
   is a co-cycle, with coefficients in $\pi_r(G_{\nu+1})$,
2. If $C^{r+1} \sim 0$, then $S(K^n)$ is simple in K^{r+1}. For this is obviously so if

$C^{r+1} = 0$. If $C^{r+1} \neq$ but ~ 0, then the coördinate system n may be replaced by one for which the corresponding co-cycle vanishes.

Since $S(K^n)$ is orientable by hypothesis, it follows that it is simple in K^1, and we have the theorem:

THEOREM 2. *If the $(r + 1)$-dimensional co-homology group of K^n vanishes for each $r = 1, \cdots, n - 1$, with coefficients in $\pi_r(G_{\nu+1})$, then any orientable sphere-space $S(K^n)$, in which the associated spheres are ν-dimensional $(\nu > 0)$, is simple.*

If the 1-dimensional co-homology group of K^n vanishes with integral coefficients, reduced mod 2, then any sphere-space $S(K^n)$ is orientable. Also the co-homology groups vanish with all coefficients if they all vanish with integral coefficients. Hence we have the corollary:

COROLLARY. *If all the co-homology groups of K^n vanish, with integral coefficients, then any sphere space $S(K^n)$ is simple.*

Notice, on the other hand, that no condition is imposed on the $(r + 1)$-dimensional cohomology groups for those values of r such that $\pi_r(G_{\nu+1}) = 0$. Do you know if there are any, beyond $r = 2$, for any $\nu > 1$?

We also have, for the reasons indicated in my paper on homotopy types:

THEOREM 3. *If a differentiable n-dimensional manifold admits an absolute parallelism, then its normal sphere-space in R^{2n+k} $(k > 0)$ is simple.*

In the paper on homotopy types I was interested only in finite (i.e. closed or bounded) manifolds. But this theorem is obviously true in general, provided one requires the manifold to be a closed, but not necessarily compact, sub-set of R^{2n+k}.

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REGULAR CYCLES OF COMPACT METRIC SPACES

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Introduction

It has long been remarked that Vietoris cycles as originally defined are not entirely adequate for expressing the connectivity structure of a space. Vietoris noted this in his initial paper[1] when he pointed out that a solenoid in euclidean 3-space possesses no non-bounding convergent 1-cycles (integer coefficients) though it can be linked by 1-cycles of the complement. On the surface there appear to be two possible reasons for this inadequacy: (1) the condition that a cycle converge is too strong, so that there are too few cycles, (2) the condition that a convergent cycle bound is too weak, so that too many cycles bound.

Pontrjagin solved the difficulty in one way by introducing compact coefficient groups.[2] The compactness enabled him to force convergence. This provided him with a sufficiency of cycles to permit a complete generalization of the Alexander duality theorem to compact spaces. An earlier noteworthy attempt to remedy the situation were the *true cycles* of Alexandroff.[3] Here no convergence requirement was made. Enough cycles of this type exist to prove theorems on dimension. However the success of the Pontrjagin method has led to their neglect. More recently Lefschetz has considered a new type called *projection cycle*.[4] The condition that such a cycle bounds is very stringent. They appear therefore to answer objection (2).

In this paper we propose a new type of cycle. In essence it is a single infinite cycle (in the compact metric space X) with the (regularity) requirement that the diameters of successive simplexes shall converge to zero. These cycles lead to homology groups $H^q(X)$ which are new topological invariants. The Vietoris homology group of one less dimension $V^{q-1}(X)$ proves to be a homomorphic image of $H^q(X)$. Explicitly, to each Vietoris cycle there corresponds a regular cycle of one higher dimension and conversely. However to certain bounding Vietoris cycles there correspond non-bounding regular cycles. In this way we succeed in strengthening the requirements for bounding, and thus answer objection (2) above. An element of the kernel $\tilde{H}^q(X)$ of the homomorphism $H^q \to V^{q-1}$ proves to be representable as the sum of the components of a true

[1] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume*, Math. Ann., 97 (1927), 454–472.

[2] L. Pontrjagin, *The general topological theorem of duality for closed sets*, Annals of Math., 35 (1934), 904–914.

[3] P. Alexandroff, *Dimensionstheorie*, Math. Ann., vol. 106, pp. 161–238.

[4] S. Lefschetz, *On chains of a topological space*, Annals of Math., 39 (1938), 383–396.

cycle. In this way true cycles reappear in a new light. Since true cycles were constructed to answer objection (1), it would seem that regular cycles answer both.

There is a "duality theorem" for the regular cycles of a closed set X in an n-sphere S^n. If $K = S^n - X$ is an infinite complex, and $H^q(K)$ is the homology group of the infinite q-cycles of K, then $H^q(K) = H^q(X)$. Thus the groups $H^q(K)$ are now proved to be topological invariants of X. It is important in this result that $H^q(K)$ is the difference group $Z^q - B^q$ where Z^q is the group of infinite cycles, and B^q is the subgroup of bounding cycles—not the closure of this group. The difference $\bar{B}^q - B^q$ is precisely the subgroup $\tilde{H}^q(X)$ of $H^q(X)$. The definition $H^q(K) = Z^q - \bar{B}^q$ has been accepted as standard in the past since this led to a topologized homology group isomorphic to $V^{q-1}(X)$. However the untopologized groups $Z^q - B^q$ have recently been shown by Eilenberg[5] to be closely related to the homotopy properties of $S^n - X$. The results of Eilenberg led to this investigation.

The essentially new invariant $\tilde{H}^q(X)$ proves to be zero whenever X is sufficiently simple, e.g. whenever X is locally connected in all dimensions in the sense of homology. It is likewise zero whenever the coefficient group is compact. However if X is a solenoid and integer coefficients are used, $\tilde{H}^1(X)$ has the power of the continuum. This result together with those of Eilenberg show that the complement of a solenoid in a 3-sphere admits essential maps on a 2-sphere.

A topology can be introduced in the group $H^q(X)$ (for example, through its isomorphism with $H^q(K)$). This topology is not very satisfactory since a point is not closed. The zero has $\tilde{H}^q(X)$ as its closure. For this reason we regard $H^q(X)$ only from the algebraic point of view throughout.

I. The homology group of the infinite cycles of a compact metric space

In this section we shall define new groups associated with a compact metric space X. For each coefficient group $\mathfrak{G}$ and each dimension q, we shall define a group $H^q(X)$ called the homology group of the regular q-cycles of X. All complexes considered will be simplicial, composed of a countable number of simplexes, and the star of any vertex shall be finite. A q-chain of a complex K is a function defined over the q-simplexes of K with values in the abelian group $\mathfrak{G}$. In case all but a finite number of coefficients of the chain are zero we shall call it a *finite* q-chain. The notions of boundary and cycle are as usual. The boundary operator is denoted by F.

Definition 1. A *regular map* of a complex K in X is a function f defined over the vertices of K with values in X such that, for any $\epsilon > 0$, all but a finite number of simplexes have their vertices imaging into sets of diameter $< \epsilon$.

Definition 2. A *regular q-chain* of X is a set of three objects: a complex A, a regular map f of A in X, and a q-chain C^q of A. In symbols: (A, f, C^q). If

[5] S. Eilenberg, *Cohomology and continuous mappings*, these Annals, 41 (1940), 231–251.

C^q is a q-cycle, (A, f, C^q) is called a *regular q-cycle* of X. We shall omit the word *regular* since all chains considered will be of this type.

DEFINITION 3. Two q-cycles (A_1, f_1, C_1^q), (A_2, f_2, C_2^q) of X are *homologous* (written $\sim$) if there exists a $(q+1)$-chain (A, f, C^{q+1}) such that A_1, A_2 are closed (not necessarily disjoint) subcomplexes of A, f agrees with f_1 on A_1 and f_2 on A_2, and $FC^{q+1} = C_1^q - C_2^q$.

The relation $\sim$ is symmetric since $F(-C^{q+1}) = C_2^q - C_1^q$. It is reflexive; for, if (A, f, C^q) is a q-cycle, the q-chain $(A, f, 0)$ is such that $F(0) = C^q - C^q$. Finally the relation is transitive; for, if (A_i, f_i, C_i^q) $(i = 1, 2, 3)$ are three cycles, the first two being homologous by virtue of the chain $(A_{12}, f_{12}, C_{12}^{q+1})$, the second two by virtue of $(A_{23}, f_{23}, C_{23}^{q+1})$, we can form the complex A composed of A_{12} and A_{23} matched in their common part A_2, and define a function f equal to f_{12} on A_{12} and f_{23} on A_{23}, then the first and third cycles are homologous by virtue of the chain $(A, f, C_{12}^{q+1} + C_{23}^{q+1})$.

These results show that the q-cycles of X divide up into mutually exclusive homology classes; two cycles are in the same class if they are homologous.

DEFINITION 4. If (A_1, f_1, C_1^q), (A_2, f_2, C_2^q) are two chains of X, their *sum* is the chain $(A_1 + A_2, f, C_1^q + C_2^q)$ where $A_1 + A_2$ is the logical sum without identifications, and f is f_1 on A_1 and f_2 on A_2.

LEMMA 1. *The sum of two homology classes of regular q-cycles is uniquely defined as the homology class containing the sum of any two representative cycles.*

If $(A_i, f_i, C_i^q) \sim (A_i', f_i', C_i^{q\prime})$ by virtue of (B_i, g_i, C_i^{q+1}) where $i = 1, 2$, then the sum $(B_1 + B_2, g, C_1^{q+1} + C_2^{q+1})$ effects the homology $\sum_{i=1}^2 (A_i, f_i, C_i^q) \sim \sum_{i=1}^2 (A_i', f_i', C_i^{q\prime})$. This shows that the homology class of the sum does not depend on the representative cycles chosen. This proves the lemma.

THEOREM 1. *The addition of homology classes defined in Lemma* 1 *satisfies the axioms for a commutative group. The group $H^q(X)$ so defined is a topological invariant of X and is called the homology group of the regular q-cycles of X.*

Inspection of Definition 4 shows that addition of homology classes is both commutative and associative. All cycles of the form (A, f, C^q) where $C^q = 0$ are in one class; for, if two such are given, the $(q+1)$-chain $(A_1 + A_2, f, C^{q+1})$ where $C^{q+1} = 0$ effects the desired homology. This class is the zero element of the group. For, if $(A_1, f_1, 0)$ is such a cycle and (A_2, f_2, C^q) is an arbitrary cycle, their sum $(A_1 + A_2, f, C^q) \sim (A_2, f_2, C^q)$ by virtue of the $(q+1)$-chain $(A_1 + A_2, f, 0)$. Finally the negative of the class containing (A, f, C^q) is the class containing $(A', f', -C^{q\prime})$ where A', f', $C^{q\prime}$ form a system isomorphic to A, f, C^q. For the complex $A + A'$ can be regarded as the two ends of the product complex of A with a line segment I, $A \times I$ can be simplicially subdivided without introducing any vertices[6] in addition to those of A and A', the map g

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[6] Order the vertices of A in a sequence $\{V^n\}$. Let $V_0^n = V^n \times 0$, $V_1^n = V^n \times 1$. If $\sigma = V^{i_0}V^{i_1} \cdots V^{i_q}$, $i_0 < i_1 < \cdots < i_q$, is a simplex of A, we introduce in $A \times I$ the simplexes $V_0^{i_0} \cdots V_0^{i_k}V_1^{i_k} \cdots V_1^{i_q}$, $k = 0, 1, \cdots, q$, together with their faces. The prism $\sigma \times I$ has then been subdivided into the chain $\sum_{k=0}^q (-1)^k V_0^{i_0} \cdots V_0^{i_k}V_1^{i_k} \cdots V_1^{i_q}$.

equal to f on A and f' on A' is regular on $A \times I$, and $C^q + (-C^{q\prime}) \sim 0$ in $A \times I$. Thus all the group axioms hold.

Due to the compactness of X the notion of a regular map of a complex in X is a topological notion. The topological invariance of $H^q(X)$ follows immediately.

THEOREM 2. *If X and Y are compact metric spaces and g is a continuous map of X and Y, then g transforms a cycle (A, f, C^q) of X into the cycle (A, gf, C^q) of Y and thereby induces a homomorphism of $H^q(X)$ into $H^q(Y)$. If g_0 and g_1 are homotopic maps of X in Y the induced homomorphisms are the same.*

The first statement is immediate since the indicated cycle transformation carries sum into sum and homologous cycles into homologous cycles.

To prove the second part it suffices to show, for any cycle (A, f, C^q) that $(A, g_0 f, C^q) \sim (A, g_1 f, C^q)$. Let $g(x, t)$, $(0 \leqq t \leqq 1)$, be a function effecting the homotopy of g_0 into g_1. Let $A \times I$ be the product of A with the unit interval $I = [0, 1]$. For each vertex $a \epsilon A$ we map the edge $a \times I$ according to the rule $\phi(a, t) = g(f(a), t)$. Then ϕ agrees with $g_0 f$ in $A \times 0$ and $g_1 f$ in $A \times 1$. We now define a subdivision of $A \times I$ into a simplicial complex so that ϕ will be a regular map. Order the vertices of A in a sequence $\{a^n\}$. On the edge $a^n \times I$ introduce the new vertices $b_s^n = (a^n, s/n + 1)$, $(s = 1, 2, \cdots, n)$. If $\sigma^p = a^{i_0} \cdots a^{i_p}$ is a simplex of A where $i_0 < i_1 < \cdots < i_p$, we set $b_s^{i_k} = (a^{i_k}, 1)$, $(s = i_k + 1, \cdots, i_p)$. Introduce in the prism $\sigma^p \times I$ the new p-simplexes $b_s^{i_0} b_s^{i_1} \cdots b_s^{i_p}$ $(s = 1, 2, \cdots, i_p)$. These divide the prism into $i_0 + 1$ non-degenerate prisms and $i_p - i_0$ degenerate ones. The map ϕ as defined over the vertices of this subdivision of $A \times I$ into a cell complex is clearly a regular map. To complete the proof we have only to subdivide again without introducing new vertices[6] so that $A \times I$ is simplicial.

To appreciate the following definition observe that, if C^q is a finite cycle of A and f is regular then $(A, f, C^q) \sim 0$; for, we may add to A the join of C^q with a new vertex P and define $f(P)$ arbitrarily, then f is still regular and $C^q \sim 0$ in the larger complex.

DEFINITION 5. A cycle will be called a *weakly bounding cycle* if it is homologous to a cycle (A, f, C^q) such that C^q is the sum of a countable number of finite cycles of A.

Any bounding cycle is weakly bounding; for, if C^q is the boundary of C^{q+1} in A, it is the sum of the boundaries of the individual terms of C^{q+1}. It is likewise clear that the sum of two weakly bounding cycles is a weakly bounding cycle.

DEFINITION 6. The subgroup $\tilde{H}^q(X)$ of $H^q(X)$ composed of classes of weakly bounding cycles is called the q^{th} *weak homology group of X.*

The following results are immediate.

THEOREM 3. *The weak homology groups of X are topological invariants of X. A homomorphism $H^q(X) \to H^q(Y)$ induced by a continuous map g of X in Y sends $\tilde{H}^q(X)$ into $\tilde{H}^q(Y)$; and this homomorphism is invariant under a homotopic deformation of g.*

Remark. Note that $H^0(X) = 0$ for every X. Suppose (A, f, C^0) is a 0-cycle. Let A^0 be the set of vertices of A, and form the product of A^0 with the half line L: $0 \leqq x < \infty$. Clearly C^0 bounds in $A^0 \times L$. If V is a vertex of A, set $f(V, x) = f(V)$ for $0 \leqq x < \infty$. Then f is a regular map of $A + A^0 \times L$.

II. Relation of a closed set to its complement in n-space

The object of this section is to prove our principal result.

Theorem 4. *If Σ is a closed n-cell, X a closed set contained in the interior of Σ, and K a realization of $\Sigma - X$ as an infinite simplicial complex, then the homology groups of the infinite q-cycles of K and of the regular q-cycles of X are isomorphic.*

The proof depends on the following notions.

Definition. If Y is a metric space and X a compact subset, a regular map f of the vertices of a complex A in Y is said to be *regular relative to* X if, for each $\epsilon > 0$, dist $[f(V), X] < \epsilon$ for almost all vertices V of A. In case f is regular relative to X the chain (A, f, C^q) is said to be *regular relative to* X. Clearly (A, f, FC^q) is likewise regular relative to X, so homology groups may be defined.

Theorem 5. *The homology group of the infinite q-cycles regular relative to X of Y is isomorphic to the homology group of the regular q-cycles of X.*

Let ϕ be a function which assigns to each point of Y one of the points of X nearest to it. If f is a regular map relative to X of A in Y, it is easy to see that ϕf is a regular map of A in X. Thus to each cycle (A, f, C^q), regular relative to X of Y, is attached the cycle $(A, \phi f, C^q)$ of X. These two cycles are homologous relative to X in Y; for the map of the product $A \times I$ of A with a line segment $I = [0, 1]$ which is f on $A \times 0$ and ϕf on $A \times 1$ is regular relative to X. Next any regular map f of A in X is a regular map relative to X of A in Y. Therefore ϕ does not change the homology class to which a cycle regular relative to X belongs. Finally, if the cycle (B, g, C^q) of X bounds the chain (A, f, C^{q+1}) regular relative to X of Y, it also bounds the chain $(A, \phi f, C^{q+1})$ of X. These facts suffice to prove the theorem.

We can now give the proof of Theorem 4. We first apply Theorem 5 using $Y = \Sigma$. Now observe that the identity map ψ of K is regular relative to X. The map ψ therefore induces a homomorphism of the homology group of the infinite q-cycles of K into the same group of Σ relative to X. To establish that this is an isomorphism we have but two things to prove:

(1) if a cycle $(K, \psi, C^q) \sim 0$ relative to X in Σ then $C^q \sim 0$ in K,

(2) every cycle regular relative to X in Σ is homologous relative to X in Σ to a cycle of the form (K, ψ, C^q).

Both of these will be proved by an application of the Veblen-Alexander deformation process.

Let A be a complex and f a regular map relative to X of A in Σ. Let $\bar{A}$ be a geometric realization of A and let $\bar{f}$ be the linear extension of the map f of A in Σ to all of $\bar{A}$. Let L be the set of points of $\bar{A}$ mapped by $\bar{f}$ into X. Since L is closed, a succession of subdivisions of $\bar{A}$ leads to a subdivision of $\bar{A} - L$ into an infinite complex B. Then $\bar{f}(B) \subset K$ and is a regular map

relative to X. We may suppose moreover that the subdivision of B is so fine that the Veblen-Alexander deformation can be applied to give a simplicial approximation g to f of B in K. Set $g = f$ on L so that g is continuous on $\bar{A}$.

To prove (1) suppose (K, ψ, C^q) bounds the chain (A, f, C^{q+1}) of Σ relative to X. Then $A \supset K$ and $\bar{f} = \psi$ on K. The subdivision of $\bar{A}$ replaces C^q, C^{q+1} by their subdivisions C_1^q, C_1^{q+1} and since g is a simplicial approximation to the identity on K, it carries C_1^q back into C^q and carries C_1^{q+1} into a chain of K whose boundary is C^q.

To prove (2) suppose (A, f, C^q) is a cycle relative to X of Σ. Let I be the interval [0, 1] and map the product $\bar{A} \times I$ in Σ by letting the map h be $\bar{f}$ on $\bar{A} \times 0$ and g on $\bar{A} \times 1$ and linear between. The open set $\bar{A} \times I - L \times 1$ may be subdivided into a simplicial complex without touching $\bar{A} \times 0$ and so that $\bar{A} \times 1 - L \times 1$ is the complex $B \times 1$. Then $C^q \times 0$ is homologous in $\bar{A} \times I - L \times 1$ to its subdivision C_1^q in $B \times 1$. Since the map h of $\bar{A} \times I - L \times 1$ is regular relative to X, we have that $(A, f, C^q) \sim (B, g, C_1^q) \sim (K, \psi, g(C_1^q))$.

It is of interest to determine the group of K that corresponds to $\tilde{H}^q(X)$ under the isomorphism of Theorem 4. If a cycle of K is a sum of finite cycles, then clearly it is a weakly bounding cycle regular relative to X. Conversely suppose (A, f, C^q) is a regular cycle of X, and $C^q = \sum C_i^q$ where C_i^q is finite. Let $\{K_i\}$ $(K_1 = \Sigma)$ be a decreasing sequence of closed polyhedral neighborhoods of X whose intersection is X and such that the boundary of K_i is a subcomplex K_i' of K. We may suppose that K_i is a finite complex under a subdivision agreeing with that of K_i', and that the mesh ϵ_i of this subdivision $\to 0$ as $i \to \infty$. Extend f linearly to a map $\bar{f}$ of $\bar{A}$ in Σ. By adding successive C_i^q and renumbering, we can arrange that $\bar{f}(C_i^q) \subset K_i$. Deform $\bar{f}(C_i^q)$ homotopically into a simplicial approximation $g(C_i^q)$. The associated deformation chain C_i^{q+1} has mesh $\leqq 2\epsilon_i +$ mesh C_i^q. Since $g(C_i^q)$ bounds in Σ, it is homologous in K_i to a cycle D_i^q of the complex $K_i' : FD_i^{q+1} = g(C_i^q) - D_i^q$. Then $D^q = \sum D_i^q$ is a cycle of K. Since D_i^{q+1} has mesh ϵ_i, $A + \sum (D_i^{q+1} + C_i^{q+1})$ is mapped regularly relative to X. Thus $(A, f, C^q) \sim (K, \psi, D^q)$ and we have proved:

Corollary. *Under the isomorphism of Theorem* 4, $\tilde{H}^q(X)$ *corresponds to the homology group of those infinite q-cycles of K which are sums of finite cycles.*

III. Relation to the Vietoris Group

The result of the last section together with known duality theorems show that, in case the coefficient group $\mathfrak{G}$ is compact, the group $H^q(X)$ of the regular q-cycles is isomorphic with the Vietoris[1] group $V^{q-1}(X)$ of dimension $q - 1$; for each is the character group of the $(n - q)$-homology group of $\Sigma - X$ based on finite cycles with coefficients from the character group $\mathfrak{G}^*$ of $\mathfrak{G}$. However when $\mathfrak{G}$ is not compact this is not always the case. The exact relation is embodied in

Theorem 6. *For an arbitrary coefficient group* $\mathfrak{G}$ *the Vietoris group of* X *of dimension* $q - 1$ *is isomorphic to the difference group of the homology group of the*

regular q-cycles by the homology group of the weakly bounding q-cycles: $V^{q-1}(X) = H^q(X) - \tilde{H}^q(X)$.

To the regular q-cycle (A, f, C^q) we attach a Vietoris $(q - 1)$-cycle Γ^{q-1} as follows. Let $\{A_i\}$, $i = 1, 2, \cdots$, be an expanding sequence of open finite subcomplexes of A whose sum is A. Let C_i^q be the part of C^q in A_i. Let $\bar{A}_i$ be the closure of A_i and set $\gamma_i = (\bar{A}_i - A_i, f, FC_i^q)$. Since the sum of the A_i is A and f is regular, for all sufficiently great i, γ_i is an ϵ-cycle of dimension $q - 1$ of X. Since $FC_i^q \sim FC_{i+1}^q$ on $\bar{A}_{i+1} - A_i$, for a given $\delta > 0$, γ_i is δ-homologous to γ_{i+1} for almost all i. Therefore $\Gamma^{q-1} = \{\gamma_i\}$ is a Vietoris cycle.

If we pass to a subsequence of the A_i the new Vietoris cycle is a subsequence of the old and therefore homologous to it. If A is expressed in some other way as the sum of an expanding sequence $\{A_i'\}$ of open subcomplexes a subsequence of this and a subsequence of the first can be interlocked to form a third expanding sequence. This shows that the homology class of Γ^{q-1} does not depend on the choice of the sequence A_i and is therefore uniquely determined by (A, f, C^q).

Now suppose (A_1, f_1, C_1^q), (A_2, f_2, C_2^q) are homologous by virtue of the chain (B, f, C^{q+1}). Express B as the sum of an expanding sequence $\{B_i\}$. Then $FC_i^{q+1} = C_{1i}^q - C_{2i}^q + D_i^q$ where D_i^q is in $\bar{B}_i - B_i$. Therefore $FD_i^q = FC_{2i}^q - FC_{1i}^q$. Since, for $\epsilon > 0$ and almost all i, $(\bar{B}_i - B_i, f, D_i^q)$ is an ϵ-chain the Vietoris cycles Γ_1^{q-1}, Γ_2^{q-1} are homologous. Thus to each element of $H^q(X)$ is assigned an element of the Vietoris group $V^{q-1}(X)$. This is a homomorphism; for the sum cycle $(A_1 + A_2, f, C_1^q + C_2^q)$ corresponds to $\{\gamma_{1i} + \gamma_{2i}\}$ if we choose $A_i = A_{1i} + B_{2i}$.

We now show that each weakly bounding regular cycle is assigned to a bounding Vietoris cycle. Suppose $C^q = \sum_1^\infty C_k^q$ where C_k^q is a finite cycle of A. Choose an $n(i)$ such that A_i does not meet $\sum_{n+1}^\infty C_k^q$. Then $\sum_1^n C_k^q = C_i^q - D_i^q$ where D_i^q lies outside A_i. Choose a j such that $\bar{A}_j \supset D_i^q$. Then $(\bar{A}_j - A_i, f, D_i^q)$ is a chain of X whose boundary is γ_i. Since f is regular, this is an ϵ-chain for a given $\epsilon > 0$ and all sufficiently great i. Therefore $\Gamma^{q-1} \sim 0$.

Conversely suppose the Vietoris cycle Γ^{q-1}, corresponding to the cycle (A, f, C^q), is bounding. Then γ_i bounds an ϵ_i-chain (B_i, ϕ_i, D_i^q) where $\epsilon_i \to 0$. Let $B = A + \sum B_i$ where for each i we identify the simplexes of FC_i^q in A with those of FD_i^q in B_i. Extend f to all of B by setting it equal to ϕ_i on B_i. Then f is a regular map of B in X. Let $\bar{C}_1^q = C_1^q - D_1^q$ and $\bar{C}_i^q = C_i^q - C_{i-1}^q - D_i^q + D_{i-1}^q$ for $i > 1$. Since $FC_i^q = FD_i^q$, $\bar{C}_i^q$ is a finite cycle of B. Since $\bar{C}_i^q$ lies outside $A_{i-1} + \sum_1^{i-1} B_k$, the infinite sum $\sum \bar{C}_i^q$ exists. Finally since $\sum_1^n \bar{C}_i^q = C_n^q - D_n^q$ we have $C^q = \sum_1^\infty \bar{C}_i^q$. Therefore $(A, f, C^q) \sim (B, f, C^q)$ is a weakly bounding cycle.

Now let $\Gamma^{q-1} = \{\gamma_i\}$ be an arbitrary Vietoris cycle; we shall find a cycle (A, f, C^q) corresponding to it. Let (A_i, f_i, D_i^q) be a δ_i-chain effecting the δ_i-homology $\gamma_{i-1} \sim \gamma_i$. Let D_0^q be the join of γ_1 with a new vertex V_0 and let A_0 be the complex carrying D_0^q. Let $A = \sum_0^\infty A_i$ where A_i is identified with A_{i-1} along their common part γ_{i-1}. Let $f = f_i$ on A_i for $i > 0$ and define

$f(V_0)$ arbitrarily in X. Then, since $\delta_i \to 0, f$ is a regular map and $(A, f, \sum_0^\infty D_i^q)$ is a q-cycle whose corresponding Vietoris $(q-1)$-cycle is Γ^{q-1}. This completes the proof of Theorem 6.

IV. Fundamental complexes

In section I we defined the homology group of the regular q-cycles of X by a procedure analogous to that of Vietoris.[1] In this section we shall parallel the procedure of Čech[7] and show that this group can be defined by using a complete family of covering complexes. The first procedure has the advantage that the groups and operations on them are obviously topologically invariant while the second procedure will be useful for computing the groups and proving theorems about their structure.

Since X is a compact metric space there exists a sequence ϕ_n, $n = 0, 1, 2, \cdots$, of finite coverings by open sets such that ϕ_n is a refinement[8] of ϕ_{n-1}, and each finite covering of X by open sets has some ϕ_n as a refinement. If ϵ_n is the maximum of the diameters of the sets of ϕ_n, this latter property insures that $\epsilon_n \to 0$. Let K_n be the complex nerve[9] of ϕ_n. For each vertex (open set) V of ϕ_n we choose some one of the vertices $V' \supset V$ of ϕ_{n-1}, define $\pi V = V'$ and obtain a simplicial map (projection) π of K_n in K_{n-1}. For convenience we shall suppose that ϕ_0 consists of the single set $V_0 = X$ so that K_0 is a vertex. We extend the definition of π by agreeing that $\pi V_0 = V_0$.

In terms of $\{K_n\}$ and π we shall construct a complex K which we shall call a fundamental complex[10] for X. The complexes K_n shall be disjoint subcomplexes of K, K_n is called the n^{th} section of K. For each n and each simplex $\sigma \in K_{n+1}$ we shall introduce in K the prismatic deformation cell $\mathfrak{D}\sigma$ of σ whose boundary is (inductively) $\pi\sigma - \sigma - \mathfrak{D}F\sigma$. We then order the vertices of K_{n+1} write the vertices $V^{i_0}, V^{i_1}, \cdots, V^{i_q}$ of σ in their natural order and subdivide $\mathfrak{D}\sigma$ simplicially[6] into the simplexes $V^{i_0} \cdots V^{i_k}\pi V^{i_k} \cdots \pi V^{i_q}$, $k = 0, 1, \cdots, q$, and their faces. In this way K is a simplicial complex. (Geometrically K is realized from the K_n by introducing between K_{n+1} and K_n line segments joining corresponding points under π).

It will be useful to extend the definition of π and $\mathfrak{D}$ to all simplexes of K. This requires that we order the vertices of K_{n+1} after we have ordered those of K_n and do this in such a way that for two vertices V, V' of K_{n+1} in the order $V < V'$ then $\pi V \leqq \pi V'$. Then, for any simplex $\sigma = V^{i_0} \cdots V^{i_k}\pi V^{i_{k+1}} \cdots \pi V^{i_r}$

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[7] E. Čech, *Théorie générale de l'homologie dans un espace quelconque*, Fund. Math., 19 (1932), 149–183.

[8] This means that each open set of ϕ_n is in some open set of ϕ_{n-1}.

[9] The vertices of K_n are the open sets of ϕ_n; a set of vertices are those of a simplex if the open sets have a non-vacuous intersection.

[10] This notion is due to Lefschetz, *Topology*, Colloquium Lectures Amer. Math. Soc., (1930), pp. 325–333.

$(i_0 < \cdots < i_k \leqq i_{k+1} < \cdots < i_r)$ of a deformation cell, we can define its projection $\pi\sigma$ to be $\pi V^{i_0} \ldots \pi V^{i_k} \pi^2 V^{i_{k+1}} \ldots \pi^2 V^{i_r}$ and

$$\mathfrak{D}\sigma = \sum_{l=0}^{k} (-1)^l V^{i_0} \ldots V^{i_l} \pi V^{i_l} \ldots \pi V^{i_r} + \sum_{l=k+1}^{r} (-1)^l \pi V^{i_0} \ldots \pi V^{i_l} \pi^2 V^{i_l} \ldots \pi^2 V^{i_r}.$$

It then follows[11] that $F\mathfrak{D}\sigma = \pi\sigma - \sigma - \mathfrak{D}F\sigma$. (Geometrically $\mathfrak{D}$ is a homotopy carrying each point of K one step nearer K_0 along the projecting lines).

Let $C^q(K)$, $Z^q(K)$, $B^q(K)$ be the groups of infinite q-dimensional chains, cycles, and bounding cycles of K with coefficients in $\mathfrak{G}$. The difference group $H^q(K) = Z^q(K) - B^q(K)$ is called the homology group of the infinite q-cycles of K. Let $\tilde{H}^q(K)$ be the subgroup of $H^q(K)$ composed of those elements representable by infinite sums of finite cycles. A cycle homologous to such a cycle we call a *weakly bounding cycle.*

THEOREM 7. *If K is a fundamental complex for the compact metric space X, then $H^q(K)$ is isomorphic with $H^q(X)$ (see Theorem 1). Under this isomorphism the subgroups $\tilde{H}^q(K)$ and $\tilde{H}^q(X)$ correspond.*

We define a regular map f of K in X as follows: if V is a vertex of K (i.e. an open set of X) we choose a point P of V and set $f(V) = P$. If a simplex σ is in the n^{th} section K_n of K, then, since diam $V \leqq \epsilon_n$ for each vertex V of K_n and the V's corresponding to the vertices of σ intersect, diam $f(\sigma) \leqq 2\epsilon_n$. By a similar argument, diam $f(\mathfrak{D}\sigma) \leqq 2\epsilon_{n-1}$. Since $\epsilon_n \to 0$ the map f is regular.

The map f assigns to each cycle C^q of K the cycle (K, f, C^q) of X. If $FC^{q+1} = C_1^q - C_2^q$, then the cycles (K, f, C_1^q) and (K, f, C_2^q) are homologous by virtue of the chain (K, f, C^{q+1}). In this way f determines a map of $H^q(K)$ in $H^q(X)$. To prove that this is a homomorphism we must show that $(K, f, C_1^q + C_2^q) \sim (K,f, C_1^q) + (K,f, C_2^q)$. The product complex $K \times I$ where I is the unit interval is mapped in a regular fashion in X by the function g identical with f on $K \times 0$ and $K \times 1$. Obviously $C_1^q \times 0 + C_2^q \times 0 \sim C_1^q \times 0 + C_2^q \times 1$ in $K \times I$ and the desired result is proved. Clearly f maps $\tilde{H}^q(K)$ into $\tilde{H}^q(X)$.

To prove that f maps only bounding cycles into such, we consider an arbitrary cycle C^q and suppose that $(K, f, C^q) \sim 0$. This means that there is a complex $K^* \supset K$, a regular extension of f to K^* and a chain C^{q+1} of K^* with C^q as boundary. We shall define a chain retraction λ of K^* in K (i.e. λ is the identity on K). For each vertex V of K we set $\lambda V = V$. Order the vertices of $K^* - K$ in a sequence and define λ stepwise as follows: λV is a vertex V' of K_n if $V' \supset f(\text{St } V)$ and $n = n(V)$ is the largest integer such that K_n contains a vertex V' with this property; if no such largest integer exists we choose an $n(V)$ one greater than that of any vertex preceding V, and a vertex V' of K_n containing $f(\text{St } V)$. For any $\sigma \in K^* - K$, let $n(\sigma)$ be the smallest of the inte-

[11] The proof of this, while lengthy, is entirely mechanical.

gers $n(V)$ associated with its vertices (if V is in K_n, $n(V) = V$). If C is any finite chain lying on and beyond the n^{th} section of K, denote by $\pi_n(C)$ the projection of C in K_n and let $\mathfrak{D}_nC$ be the associated deformation chain: $F\mathfrak{D}_nC = \pi_nC - C - \mathfrak{D}_nFC$. We extend the definition of λ to the simplexes of $K^* - K$ considering them in the order of their dimensions. Suppose λ has been defined on all simplexes of dimension $< k$ so as to satisfy the following requirements: (1) $\lambda F = F\lambda$, (2) $\lambda\sigma$ lies on and beyond $K_{n(\sigma)}$, (3) $\pi_{n(\sigma)}\lambda\sigma = \pi_{n(\sigma)}\sigma$ where the latter chain is the simplex of $K_{n(\sigma)}$ whose vertices are the projections in $K_{n(\sigma)}$ of the images of the vertices of σ in K (the existence of this simplex is assured by $\lambda V \supset f(\text{St } V)$). Consider now a simplex σ of dimension k. If σ is in K, we set $\lambda\sigma = \sigma$ and observe that (1), (2), and (3) still hold. If σ is in $K^* - K$, then $\lambda F\sigma$ is defined. For any face σ' of σ, we know that $n(\sigma') \geqq n(\sigma)$, consequently by (2) and (3) $\pi_{n(\sigma)}\lambda F\sigma = \pi_{n(\sigma)}F\sigma$. We now set $\lambda\sigma = \pi_{n(\sigma)}\sigma - \mathfrak{D}_{n(\sigma)}\lambda F\sigma$. Then $F\lambda\sigma = \pi_{n(\sigma)}F\sigma - \pi_{n(\sigma)}\lambda F\sigma + \lambda F\sigma = \lambda F\sigma$, so that (1) holds. Property (2) holds by definition. Since any deformation cell of K projects into zero under π_n, we obtain $\pi_{n(\sigma)}\lambda\sigma = \pi^2_{n(\sigma)}\sigma = \pi_{n(\sigma)}\sigma$, so (3) holds. Since f is a regular map of K^* in X, for only a finite number of vertices can $n(V)$ be less than a fixed integer. Since K^* is locally finite and since (2) holds, it follows that any simplex of K is contained in the image of at most a finite number of simplexes of K^*. Thus λ is a true chain retraction of K^* into K. Since $\lambda F = F\lambda$, we obtain $F\lambda C^{q+1} = C^q$. This completes the proof that f maps only bounding cycles into such.

To complete the proof of Theorem 4 we must show that any cycle (A, g, C^q) is homologous to a cycle of the form (K, f, C^q_1); and if C^q is a sum of finite cycles so also is C^q_1. Perform the construction of the last paragraph on the complex $K^* = K + A$. We shall prove that $(K, f, \lambda C^q) \sim (A, g, C^q)$ by imbedding K^* in a complex K' as follows. The vertices of K' shall be those of K^* and each simplex of K^* shall be in K'. Furthermore if $\sigma \in A$ has $U^0, U^1, \cdots, U^q$ as its vertices and $V^1, \cdots, V^k$ are the vertices of the chain $\lambda\sigma$, then $U^0, \cdots, U^q, V^1, \cdots, V^k$ shall be the vertices of a simplex of K' as also each of its faces. Let h be the map of K' in X equal to f on K and g on A. Then K' is a locally finite complex and h is a regular map. Since for each $\sigma \in A$, σ and $\lambda\sigma$ lie on a simplex of K', we can define, inductively on the dimension, a deformation operator in K' associated with the chain transformation λ. In this way $C^q \subset A$ is homologous in K' to $\lambda C^q \subset K$. If C^q is a sum of finite cycles then λC^q is the sum of the images under λ of these cycles. This completes the proof of Theorem 7.

Remark I. Theorem 7 provides an easy proof that, *if $\mathfrak{G}$ is compact, any weakly bounding cycle of X is bounding, and therefore $\tilde{H}^q(X) = 0$.* Since the group $C^{q+1}(K)$ of $(q + 1)$-chains of K is the unrestricted direct sum of a countable number of summands each isomorphic to $\mathfrak{G}$ and $\mathfrak{G}$ is compact, $C^{q+1}(K)$ is likewise compact. Since F is a continuous map of $C^{q+1}(K)$ on $B^q(K)$ the latter group is likewise compact and therefore closed in $Z^q(K)$. Now any finite cycle of K can be deformed into K_0 by a finite number of iterations of $\mathfrak{D}$, and is there-

fore bounding. Thus any sum of finite cycles is a limit of bounding cycles and therefore lies in the closure of $B^q(K)$. Since this group is closed any sum of finite cycles of K is bounding.

II. The question naturally arises as to what cohomology theory corresponds to the homology theory of $H^q(X)$? The proof of Theorem 7 shows that this question reduces to the same one for a fundamental complex K of X. The finite q-cocycles of K (coefficients in $\mathfrak{G}$) are dual to the infinite q-cycles of K (coefficients in the compact group $\mathfrak{G}^*$ of all characters of $\mathfrak{G}$). Since $\mathfrak{G}^*$ is compact, $H^q(K)$ is $V^{q-1}(X)$; and therefore *the* q^{th} *cohomology group of* K *(finite cocycles) is the usual* $(q - 1)^{\text{st}}$ *cohomology group of* X. As for infinite cocycles, *each such is the coboundary of an infinite chain.* This is seen as follows. K is homotopic to the point K_0, each point moving along the projecting lines of K. This can be made a chain homotopy so that to each simplex σ^p is assigned a deformation $(p + 1)$-chain $\mathfrak{D}\sigma^p$. The dual chain transformation attaches to every σ^p an infinite $(p - 1)$-chain $\mathfrak{D}^*\sigma^p$ (composed of those $(p - 1)$-simplexes sweeping out σ^p during the homotopy). If F^* denotes coboundary, the usual homotopy relation $F^*\mathfrak{D}^*\sigma^p = -\sigma^p - \mathfrak{D}^*F^*\sigma^p$ holds. Since $\mathfrak{D}^*\sigma^p$ follows σ^p in the natural order along K, $\mathfrak{D}^*C^p$ can be defined linearly for any infinite chain. If C^p is a cocycle then $F^*(-\mathfrak{D}^*C^p) = C^p$. It may seem surprising that this should be true of K and yet not true of the complex $\Sigma - X$ of Theorem 4. However the construction in the proof of Theorem 4 applied to K defines a chain transformation of K in $\Sigma - X$ attaching to a simplex of K a possibly infinite chain of $\Sigma - X$. Consequently, the dual chain transformation of $\Sigma - X$ in K is definable only for finite cochains of $\Sigma - X$.

V. Weakly bounding cycles

In this section our object is to show how to compute the homology group $\tilde{H}^q(X)$ of the weakly bounding cycles in terms of the coefficient group $\mathfrak{G}$ and the Vietoris group $V^q(X)$ based on real numbers mod 1 as coefficients (see Theorem 8). As a corollary it will follow that $\tilde{H}^q(X) = 0$ for a space X which is sufficiently smooth locally. Another consequence is that $\tilde{H}^q(X)$ is expressible in terms of known invariants. We have seen, (Theorem 6) that this is also true of the difference group $H^q(X) - \tilde{H}^q(X)$. We do not know whether $\tilde{H}^q(X)$ is always a direct summand of $H^q(X)$. A positive answer would mean that $H^q(X)$ is likewise expressible in terms of known invariants.

We shall suppose we have at hand a fundamental complex K for X with the operations π and $\mathfrak{D}$ as defined in section IV. Let K' be the subcomplex of K composed of the sum of the sections $K_0, K_1, \cdots, K_n, \cdots$.

LEMMA 3. *A weakly bounding cycle on* K *is homologous to a cycle on* K'. *Conversely any cycle on* K' *is weakly bounding.*

The second statement is immediate since K' is a sum of finite disjoint complexes; so any cycle on K' must therefore be a sum of finite cycles.

For the first part, let C^q be a finite cycle of K. We define λC^q as follows. Choose the first section K_n which contains a face of C^q and deform C^q along the

gers $n(V)$ associated with its vertices (if V is in K_n, $n(V) = V$). If C is any finite chain lying on and beyond the n^{th} section of K, denote by $\pi_n(C)$ the projection of C in K_n and let $\mathfrak{D}_n C$ be the associated deformation chain: $F\mathfrak{D}_n C = \pi_n C - C - \mathfrak{D}_n FC$. We extend the definition of λ to the simplexes of $K^* - K$ considering them in the order of their dimensions. Suppose λ has been defined on all simplexes of dimension $< k$ so as to satisfy the following requirements: (1) $\lambda F = F\lambda$, (2) $\lambda\sigma$ lies on and beyond $K_{n(\sigma)}$, (3) $\pi_{n(\sigma)}\lambda\sigma = \pi_{n(\sigma)}\sigma$ where the latter chain is the simplex of $K_{n(\sigma)}$ whose vertices are the projections in $K_{n(\sigma)}$ of the images of the vertices of σ in K (the existence of this simplex is assured by $\lambda V \supset f(\text{St } V)$). Consider now a simplex σ of dimension k. If σ is in K, we set $\lambda\sigma = \sigma$ and observe that (1), (2), and (3) still hold. If σ is in $K^* - K$, then $\lambda F\sigma$ is defined. For any face σ' of σ, we know that $n(\sigma') \geqq n(\sigma)$, consequently by (2) and (3) $\pi_{n(\sigma)}\lambda F\sigma = \pi_{n(\sigma)}F\sigma$. We now set $\lambda\sigma = \pi_{n(\sigma)}\sigma - \mathfrak{D}_{n(\sigma)}\lambda F\sigma$. Then $F\lambda\sigma = \pi_{n(\sigma)}F\sigma - \pi_{n(\sigma)}\lambda F\sigma + \lambda F\sigma = \lambda F\sigma$, so that (1) holds. Property (2) holds by definition. Since any deformation cell of K projects into zero under π_n, we obtain $\pi_{n(\sigma)}\lambda\sigma = \pi^2_{n(\sigma)}\sigma = \pi_{n(\sigma)}\sigma$, so (3) holds. Since f is a regular map of K^* in X, for only a finite number of vertices can $n(V)$ be less than a fixed integer. Since K^* is locally finite and since (2) holds, it follows that any simplex of K is contained in the image of at most a finite number of simplexes of K^*. Thus λ is a true chain retraction of K^* into K. Since $\lambda F = F\lambda$, we obtain $F\lambda C^{q+1} = C^q$. This completes the proof that f maps only bounding cycles into such.

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V. Weakly bounding cycles

In this section our object is to show how to compute the homology group $\tilde{H}^q(X)$ of the weakly bounding cycles in terms of the coefficient group $\mathfrak{G}$ and the Vietoris group $V^q(X)$ based on real numbers mod 1 as coefficients (see Theorem 8). As a corollary it will follow that $\tilde{H}^q(X) = 0$ for a space X which is sufficiently smooth locally. Another consequence is that $\tilde{H}^q(X)$ is expressible in terms of known invariants. We have seen, (Theorem 6) that this is also true of the difference group $H^q(X) - \tilde{H}^q(X)$. We do not know whether $\tilde{H}^q(X)$ is always a direct summand of $H^q(X)$. A positive answer would mean that $H^q(X)$ is likewise expressible in terms of known invariants.

We shall suppose we have at hand a fundamental complex K for X with the operations π and $\mathfrak{D}$ as defined in section IV. Let K' be the subcomplex of K composed of the sum of the sections $K_0, K_1, \cdots, K_n, \cdots$.

LEMMA 3. *A weakly bounding cycle on K is homologous to a cycle on K'. Conversely any cycle on K' is weakly bounding.*

The second statement is immediate since K' is a sum of finite disjoint complexes; so any cycle on K' must therefore be a sum of finite cycles.

For the first part, let C^q be a finite cycle of K. We define λC^q as follows. Choose the first section K_n which contains a face of C^q and deform C^q along the

projection lines of K into K_n; the resulting cycle of K_n is λC^q. Now given any sum $\sum C_i^q$ of finite cycles the sum $\sum \lambda C_i^q$ likewise exists, is homologous to it, and lies on K'.

LEMMA 4. *If a cycle* Γ *of* K' *bounds on* K, *it is homologous in* K' *to a cycle of the form* $\pi\Gamma' - \Gamma'$ *where* Γ' *is in* K'. *Conversely any cycle of this type bounds in* K.

Since $\pi\Gamma' \sim \Gamma'$ in K, the second part is immediate. For the first part, write $\Gamma^q = \sum \gamma_n^q$ where $\gamma_n^q \subset K_n$. Let $\bar{K}$ denote the cell complex composed of K' and the deformation cells of simplexes of K' so that K arises from $\bar{K}$ on subdividing simplicially the deformation cells. Since Γ^q bounds in K, it bounds a chain Δ^{q+1} of $\bar{K}$. Write Δ^{q+1} in the form $\sum (\Delta_n^{q+1} + \mathfrak{D}\Lambda_n^q)$ where Δ_n^{q+1}, Λ_n^q are in K_n. Since Δ^{q+1} has no boundary between K_n and K_{n-1}, Λ_n^q must be a cycle. Since γ_n^q is the part of $F\Delta^{q+1}$ in K_n we have $\gamma_n^q = F\Delta_n^{q+1} + \pi\Lambda_{n+1}^q - \Lambda_n^q$. Set $\Gamma^{q\prime} = \sum \Lambda_n^q$; and we obtain $\Gamma^q \sim \pi\Gamma^{q\prime} - \Gamma^{q\prime}$ in K'.

The last two lemmas show how to reduce the computation of $\tilde{H}^q(X)$ to one involving only K' and π. We put these results together in

LEMMA 5. *The group* $\tilde{H}^q(X)$ *is isomorphic to the difference group* $H^q(K') - L^q(K')$ *where* $L^q(K')$ *is the subgroup of elements of the form* $\pi x - x$ *where* $x \in H^q(K')$.

Our immediate object now is to show that the torsion cycles of K' all belong to $L^q(K')$. We obtain this after a few preliminaries.

LEMMA 6. *If* Γ *is a cycle of* K' *and* $\pi\Gamma$ *is homologous in* K' *to a cycle of the form* $\pi\Gamma' - \Gamma'$, *then this is also true of* Γ.

This follows from the computation

$$\Gamma = \pi\Gamma - (\pi\Gamma - \Gamma) \sim \pi\Gamma' - \Gamma' - (\pi\Gamma - \Gamma) = \pi(\Gamma' - \Gamma) - (\Gamma' - \Gamma).$$

LEMMA 7. *If* $\Gamma = \sum \gamma_n$, $\gamma_n \subset K_n$, *is a cycle of* K' *and each* γ_n *is a component of a Čech cycle (i.e. there exist cycles* $\gamma_{ni} \subset K_i$ $(i > n)$ *such that* $\pi\gamma_{ni} \sim \gamma_{n,i-1}$ $(i > n + 1)$ *and* $\pi\gamma_{n,n+1} \sim \gamma_n$*) then* Γ *is in* $L^q(K')$.

Let $\Gamma_n = \sum_{i=n+1}^{\infty} \gamma_{ni}$. Then $\pi\Gamma_n - \Gamma_n \sim \gamma_n$ in K'. Since Γ_n lies on $\sum_{n+1}^{\infty} K_i$, the sum $\Gamma' = \sum_1^{\infty} \Gamma_n$ exists and $\Gamma \sim \pi\Gamma' - \Gamma'$.

DEFINITION. Let $T^q(K_n)$ be the torsion subgroup of the homology group $H^q(K_n)$, i.e. those elements representable as linear forms with coefficients from $\mathfrak{G}$ in torsion cycles with integer coefficients. Let $\hat{T}^q(K_n)$ be the subgroup of $T^q(K_n)$ composed of components of Čech cycles.

LEMMA 8. *To each integer* n *corresponds an integer* $k(n) > n$ *such that* $\pi^{k(n)-n}T^q(K_{k(n)}) = \hat{T}^q(K_n)$.

Consider first the special case when $\mathfrak{G}$ is the group of integers. Then $T^q(K_n)$ is a finite group. Let $\bar{T}^q(K_n)$ be the common part of the projections of $T^q(K_i)$ in $T^q(K_n)$ for $i \geqq n$. Since these projections form a decreasing sequence and $T^q(K_n)$ is finite, there is a $k(n) > n$ such that $T^q(K_{k(n)})$ projects into $\bar{T}^q(K_n)$. Then also $\bar{T}^q(K_{k(n)})$ projects into $\bar{T}^q(K_n)$, and therefore $\pi\bar{T}^q(K_{n+1}) = \bar{T}^q(K_n)$. This means that $\bar{T}^q(K_n) = \hat{T}^q(K_n)$; for, if $\gamma_n \in \bar{T}^q(K_n)$ is given, we can choose successively $\gamma_i \in \bar{T}^q(K_i)$, $(i = n + 1, n + 2, \cdots)$, such that $\pi\gamma_{i+1} = \gamma_i$.

Consider now an arbitrary coefficient group $\mathfrak{G}$. Let $\bar{T}^q(K_n, \mathfrak{G})$ be the subgroup of those elements of $T^q(K_n, \mathfrak{G})$ representable as linear forms with co-

efficients in $\mathfrak{G}$ in torsion cycles with integer coefficients from $\hat{T}^q(K_n)$. Then $T^q(K_{k(n)}, \mathfrak{G})$ projects into $\bar{T}^q(K_n, \mathfrak{G})$. Furthermore $\pi\bar{T}^q(K_{n+1}, \mathfrak{G}) = \bar{T}^q(K_n, \mathfrak{G})$; for, if $\sum g_i\gamma_i \in \bar{T}^q(K_n, \mathfrak{G})$ is given, $g_i \in \mathfrak{G}$, $\gamma_i \in \hat{T}^q(K_n)$, choose $\gamma_i' \in \hat{T}^q(K_{n+1})$ such that $\pi\gamma_i' = \gamma_i$, then $\pi\sum g_i\gamma_i' = \sum g_i\gamma_i$. It now follows as above that $\bar{T}^q(K_n, \mathfrak{G}) = \hat{T}^q(K_n, \mathfrak{G})$, and the lemma is proved.

LEMMA 9. *The unrestricted direct sum* $\sum_{n=0}^{\infty} T^q(K_n)$ *is in* $L^q(K')$ (*see Lemma* 5).

Suppose $\gamma_n \in T^q(K_n)$, $n = 0, 1, 2, \cdots$. If $i \geqq k(n)$, then, by Lemma 8, $\pi^{i-n}\gamma_i \in \hat{T}^q(K_n)$. Therefore by Lemma 7, $\pi^{i-n}\gamma_i \sim \pi\Gamma_i - \Gamma_i$ where Γ_i lies in $\sum_{n+1}^{\infty} K_j$. Iteration of Lemma 6 shows that $\gamma_i \sim \pi\Gamma_i' - \Gamma_i'$ where Γ_i' is in $\sum_{n+1}^{\infty} K_j$. If, for a given i, we choose the largest n such that $i \geqq k(n)$ (such an n exists since $k(0) = 1$) then $\Gamma' = \sum_1^{\infty} \Gamma_i'$ exists and $\sum_1^{\infty} \gamma_i = \pi\Gamma' - \Gamma'$.

DEFINITION. Let $S^q(K_n)$ be the reduced homology group $H^q(K_n) - T^q(K_n)$. Since π maps torsion cycles into such, π induces a homomorphism of $S^q(K_{n+1})$ in $S^q(K_n)$. Furthermore $H^q(K_n)$ is[12] the direct sum $T^q(K_n) + S^q(K_n)$. Let $S^q(K')$ be the unrestricted direct sum $\sum_{n=0}^{\infty} S^q(K_n)$.

Since $H^q(K')$ is the unrestricted direct sum $\sum_{n=0}^{\infty} H^q(K_n)$, it decomposes into the direct sum $\sum T^q(K_n) + S^q(K')$. Since the first of these groups is in $L^q(K')$ we can improve Lemma 5 as follows

LEMMA 10. *The group* $\tilde{H}^q(X)$ *is isomorphic to the difference group* $S^q(K') - M^q(K')$ *where* $M^q(K')$ *is the set of those elements of* $S^q(K')$ *of the form* $\pi x - x$ *where* $x \in S^q(K')$.

DEFINITION. Let $\{H_n\}$ be a direct homomorphism sequence[13] of discrete abelian groups: $\pi H_{n-1} \subset H_n$. The dual inverse sequence relative to the abelian group $\mathfrak{G}$ is the inverse sequence $\{S_n\}$, $\pi S_n \subset S_{n-1}$, defined as follows. S_n is the group of all homomorphisms of H_n in $\mathfrak{G}$. If $s_n \in S_n$, the homomorphism πH_{n-1} followed by s_n determines a homomorphism πs_n of H_{n-1} in $\mathfrak{G}$ (i.e. $\pi s_n \in S_{n-1}$). If the image of $h_n \in H_n$ in $\mathfrak{G}$ under s_n is denoted by $s_n \cdot h_n$, then $s_n \cdot \pi h_{n-1} = \pi s_n \cdot h_{n-1}$ for every $s_n \in S_n$, $h_{n-1} \in H_{n-1}$. It follows that π maps S_n homomorphically in S_{n-1}. We note that, if $\mathfrak{G}$ is the group of reals mod 1, then $\{H_n\}$ and $\{S_n\}$ are dual in the sense of character groups.

THEOREM 8. *If X is a compact metric space the group $\tilde{H}^q(X)$ of the weakly bounding cycles (coefficients in $\mathfrak{G}$) can be computed as follows. Let $V_q(X)$ be the cohomology group of X based on integer coefficients. Let $\{V^n\}$ be any direct sequence of groups each on a finite basis having $V_q(X)$ as limit group. In addition let $V^0 = 0$. Let $\{S_n\}$ be the dual inverse sequence relative to $\mathfrak{G}$. Let $S =$ the unrestricted direct sum $\sum S_n$, and M the subgroup of elements of the form $\pi s - s$ for $s \in S$. Then $\tilde{H}^q(X)$ is the difference group $S - M$.*

If V_q^n is the q^{th} cohomology group of K_n (integer coefficients) where $\{K_n, \pi\}$ are covering complexes and projections for X, then it is well known[14] that the

[12] N. E. Steenrod, *Universal homology groups*, Amer. Jour. of Math., 58 (1936), 661–701, Theorem 8.

[13] *Loc. cit.* p. 669.

[14] *Loc. cit.* Theorem 6.1.

limit group of the direct sequence $\{V_q^n\}$ is the group $V_q(X)$. Furthermore the group $S^q(K_n)$ (coefficients in $\mathfrak{G}$) is the group of all homomorphisms of V_q^n in $\mathfrak{G}$ where the multiplication $s \cdot h$, for $s \in S^q(K_n)$, $h \in V_q^n$, is the Kronecker intersection.[15] Finally the Kronecker intersection has the property that, for any $h \in V_q^{n-1}$ and $s \in S^q(K_n)$, $s \cdot \pi h = \pi s \cdot h$. Therefore Lemma 10 asserts the truth of the theorem in case the V_q^n are cohomology groups of a complete sequence of covering complexes. We shall complete the proof by showing that the group $S - M$ does not depend on the choice of the sequence $\{V^n\}$.

Now any two sequences $\{V^n\}$, $\{V^{n'}\}$ with the same limit group V are equivalent in the sense that there exist a finite number of sequences $\{V^n\} = \{V_1^n\}$, $\{V_2^n\}, \cdots, \{V_k^n\} = \{V^{n'}\}$ such that, for any adjacent pair, one is a subsequence of the other.[16] The proof will therefore be complete if we can show that $S - M$ is the same for $\{V^n\}$ as for any subsequence $\{V^{n_k}\}$. We suppose of course that $V^{n_0} = V^0 = 0$. Let $S = \sum S_n$, $S' = \sum S_{n_k}$. Then S' is a subgroup of S. The projection π' of S_{n_k} into $S_{n_{k-1}}$ is the projection $\pi^{n_k - n_{k-1}}$. Elements of M' can be written $\pi s' - s'$ where $s' = \sum s'_{n_k}$, $s'_{n_k} \in S_{n_k}$. In terms of s' construct an element $s = \sum s_k \in S$ as follows: $s_{n_k} = s'_{n_k}$, $s_i = \pi^{n_k - i} s_{n_k}$ for $i = n_{k-1} + 1, \cdots, n_k - 1$. It then follows that $\pi s - s = \pi' s' - s'$. Therefore M' is a subgroup of M.

Suppose now that $s' = \sum s'_{n_k} \in S'$ is in M. Then there is an $s = \sum s_k$ such that $\pi s - s = s'$. Then, for $i = n_{k-1} + 2, \cdots, n_k$, we must have $\pi s_i = s_{i-1}$. Therefore $\pi s_{n_{k-1}+1} = \pi' s_{n_k}$. This gives $s'_{n_{k-1}} = \pi' s_{n_k} - s_{n_{k-1}}$. Consequently if we set $s'' = \sum s_{n_k}$ we get $\pi' s'' - s'' = s' \in M'$. Therefore S' intersects M in M'.

This means that $S' - M'$ appears as a subgroup of $S - M$. To prove that it is the entire group we will show that any element $s = \sum s_k$ of S can be reduced to an element of S' by subtracting an element of M. This requires finding an $s' = \sum s'_k$ such that $s'_i = 0$ for $i = n_k + 1, \cdots, n_{k+1} - 1$ and $s - s' \in M$. If j is an arbitrary index and $s'' = -s_j$, then $(\sum s_i) - (\pi s'' - s'')$ has a zero in the j^{th} term, is $s_{j-1} + \pi s_j$ in the $(j-1)^{\text{st}}$ and agrees with $\sum s_i$ elsewhere. If we begin with $j = n_k - 1$ and continue down to $j = n_{k-1} + 1$ with this process we obtain finally an s^{k-1} with zero coordinates outside the indicated range of j such that $s - (\pi s^{k-1} - s^{k-1})$ has zeros for this range of j. Clearly $s'' = \sum_{k=0}^{\infty} s^k$ exists, and finally $s - (\pi s'' - s'')$ is in S'. This completes the proof of Theorem 8.

THEOREM 9. *If the q^{th} cohomology group of X based on integer coefficients has a finite basis then the group $\tilde{H}^q(X) = 0$ for an arbitrary coefficient group $\mathfrak{G}$.*

In this case we can choose each V^n identical with $V_q(X)$ and each π an isomorphism (except $V^0 = 0$). This means that $\pi S_n \subset S_{n-1}$ is an isomorphism (except $\pi S_1 = 0$). It follows now exactly as in the proof of Lemma 7 that $S = M$.

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[15] H. Whitney, *On matrices of integers and combinatorial topology*, Duke Jour., 3 (1937), 35–45.

[16] See footnote 11, Theorem 7.2.

COROLLARY. *If X is locally connected in the sense of homology in the first q dimensions then $\tilde{H}^q(X) = 0$ for an arbitrary coefficient group $\mathfrak{G}$.*

It is well known that the local connectedness of X leads to the result that the homology groups of X are subgroups of those of a finite complex.

THEOREM 10. *If V^q is q^{th} homology group of X based on the reals* mod 1 *as coefficients and $\bar{V}^q$ is the component of the identity of V^q, then the group $\tilde{H}^q(X)$ is the group $\tilde{H}^1(\bar{V}^q)$ if the coefficient group $\mathfrak{G}$ has no elements of finite order.*

The q^{th} cohomology group V_q of X based on integer coefficients is the character group of V^q. Let F_q be the subgroup of elements of finite order of V_q. Then[17] $\bar{V}_q = V_q - F_q$ is the character group of $\bar{V}^q$. Let $\{V^n\}$ be a direct sequence of groups each on a finite basis, having V_q as limit group and let F^n be the subgroup of elements of finite order of V^n. Then the $\bar{V}^n = V^n - F^n$ form a direct sequence having $\bar{V}_q$ as limit group. Since $\mathfrak{G}$ has no elements of finite order, any homomorphism of V^n in $\mathfrak{G}$, maps F^n into zero and can therefore be realized as a homomorphism of $\bar{V}^n$ in $\mathfrak{G}$. Therefore S_n is the group of homomorphism of $\bar{V}^n$ in $\mathfrak{G}$. Consequently we can replace V_q by $\bar{V}_q$ in Theorem 8. Now the character group of a compact connected group is isomorphic with the 1-dimensional cohomology group of the compact group based on integer coefficients,[18] i.e. $\bar{V}_q = V_1(\bar{V}^q)$. Therefore $\tilde{H}^q(X)$ and $\tilde{H}^1(\bar{V}^q)$ are computed in the same way from isomorphic groups.

VI. Examples

The emphasis placed on the group $\tilde{H}^q(X)$ in the foregoing sections is justifiable only if it can be shown that it sometimes is not the zero group. We propose to give such an example. Our results show that the coefficient group $\mathfrak{G}$ must not be compact if $\tilde{H}^q(X)$ is not to be zero. In this example $\mathfrak{G}$ will be the group of integers. According to Theorem 10 we must seek our example among the compact connected abelian groups restricting ourselves to the 1-dimensional group $\tilde{H}^1$. According to the Corollary to Theorem 9 the group must not be locally connected. The possibilities then are definitely limited. However we have

THEOREM 11. *If X is a 1-dimensional solenoidal group then $\tilde{H}^1(X) = H^1(X)$ based on integer coefficients has a continuum of elements each of infinite order.*

Any solenoid can be defined as the limit group of an inverse homomorphism sequence $\{X_i\}$ such that each X_i is isomorphic with the group of reals mod 1, and X_i is mapped by π a_{i-1} times around X_{i-1}. We can suppose $X_0 = 0$ and $\pi X_1 = 0$. Since X is a solenoid there exist infinitely many $a_i \neq \pm 1$ and almost all are $\neq 0$. By passing to a subsequence and properly orienting each X_i we can arrange that each $a_i \geqq 2$ for $i \geqq 1$ (of course $a_0 = 0$). The 1-dimensional cohomology group $V_1(X)$ is[18] the character group of X. Hence the sequence $\{V^i\}$ (see Theorem 8) can be chosen as the direct sequence dual to $\{X_i\}$. Then

[17] L. Pontrjagin, *Topological groups*, vol. 2 of Princeton Math. Series, 1939.

[18] See footnote 11, Theorem 15.

V^i is isomorphic to the group of integers as is therefore the group S_i of homomorphisms of V^i in $\mathfrak{G}$. Choosing properly the generators $s_i \in S_i$, the homomorphism $\pi S_i \subset S_{i-1}$ is given by the relation $\pi s_i = a_{i-1}s_{i-1}$.

We are thus led to the following construction of $\tilde{H}^1(X)$. Let S be the group of all sequences $(x_1, x_2, \cdots, x_n, \cdots)$ of integers. A sequence $\{x_n\}$ is in the subgroup M if the equations

$$\text{(A)} \qquad a_n y_{n+1} - y_n = x_n \qquad (n = 1, 2, \cdots)$$

admit a solution in integers $\{y_n\}$. M is a subgroup; for, if $\{y_n\}$, $\{y'_n\}$ satisfy (A) for $\{x_n\}$, $\{x'_n\}$ respectively, then $\{y_n + y'_n\}$ satisfies (A) for $\{x_n + x'_n\}$. Then $\tilde{H}^1(X)$ is the group $S - M$.

In order to study the equations (A) define $b_{ij} = \prod_{k=i}^{k=j} a_k$ $(i \leqq j)$ and solve the first n equations for y_{n+1} in terms of $y_1, x_1, \cdots, x_n$:

$$\text{(B)} \qquad y_{n+1} = \frac{x_n}{b_{nn}} + \frac{x_{n-1}}{b_{n-1,n}} + \cdots + \frac{x_1 + y_1}{b_{1n}}.$$

Therefore, for each choice of y_1, the equations (A) admit a unique solution in real numbers $\{y_n\}$. Now let y_1, y'_1 be two distinct real numbers, then

$$y_{n+1} - y'_{n+1} = (y_1 - y'_1)/b_{1n}.$$

Since $b_{1n} \to \infty$, *any two solutions* $\{y_n\}$, $\{y'_n\}$ *of* (A) *in real numbers are asymptotic.* This has the following consequences. First, *the equations* (A) *admit at most one solution in integers* $\{y_n\}$. Secondly, *if a solution* $\{y_n\}$ *of* (A) *in real numbers is such that a subsequence* $\{y_{n_k}\}$ *converges to a number* y *not an integer, then the equations* (A) *admit no solution in integers.*

We shall use this last result to exhibit sequences $\{x_n\}$ such that (A) admits no solution in integers. Let y be real and $0 < y < 1$. We define a sequence $\{x_n\}$ to correspond to y as follows. For each integer n we define x_i for $i = (n-1)^2 + 1, \cdots, n^2$ according to following inductive rule: x_{n^2} is the largest integer such that $s_0^n = x_{n^2}/b_{n^2n^2} \leqq y$, let

$$s_i^n = \frac{x_{n^2}}{b_{n^2n^2}} + \frac{x_{n^2-1}}{b_{n^2-1,n^2}} + \cdots + \frac{x_{n^2-i}}{b_{n^2-i,n^2}}$$

and suppose $x_{n^2}, \cdots, x_{n^2-i+1}$ have been chosen so that $s_{i-1}^n \leqq y$, if $i \leqq n-1$, choose x_{n^2-i} to be the largest integer such that $s_i^n \leqq y$, if $i = n, n+1, \cdots, n^2 - (n-1)^2 - 1$, let $x_{n^2-i} = 0$. Observe now that $|\, y - s_{n-1}^n \,| < 1/b_{n^2-n+1,n^2} \leqq 2^{-n}$ (since each $a_i \geqq 2$). Furthermore, if we choose $y_1 = 0$, then (B) gives

$$y_{n^2+1} = s_{n-1}^n + \frac{s_{n-2}^{n-1}}{b_{(n-1)^2+1,n^2}} + \cdots + \frac{s_0^1}{b_{2,n^2}}.$$

Then $|\, y_{n^2+1} - s_{n-1}^n \,| \leqq y2^{-2n-2}$. Thus $\lim_{n\to\infty} y_{n^2+1} = y$. Since y is not an integer it follows that the sequence $\{x_i\}$ is such that the equations (A) cannot be solved in integers. Therefore $\{x_i\}$ is not in the subgroup M.

Next, if $0 < y' < y < 1$, and $\{x'_i\}$, $\{x_i\}$ are the sequences corresponding to y', y, then the sequence $\{x_i - x'_i\}$ is likewise not in M; for lim $(y_{n^2+1} - y'_{n^2+1}) = y - y'$ is not an integer. Thus we have established a $1 - 1$ correspondence between the numbers $0 < y < 1$ and a subset of the group $S - M$. If y is irrational, then no multiple of y is an integer; therefore no multiple of $\{x_i\}$ is in the group M. Since we can choose a continuum of y's which are rationally independent, Theorem 11 is proved.

THEOREM 12. *If X is a p-adic solenoid (i.e. each a_n = the integer p), then the group $\tilde{H}^1(X)$ has the following properties:*

(*a*) *every element is uniquely divisible by p,*

(*b*) *no element has the finite order p,*

(*c*) *a periodic sequence of period k represents an element of the finite order $p^k - 1$,*

(*d*) *if q is prime to p, there is an element of order q.*

If $\{x_n\}$ is an arbitrary element of S, let $x'_i = x_{i+1}$. Then $p\{x'_i\} - \{x_i\} = \{px_{i+1} - x_i\}$ is in M, and $\{x_i\}$ is p times $\{x'_i\}$ in $S - M$. Thus division by p is possible. The uniqueness will follow when (b) is proved.

Suppose $p\{x_n\} = \{py_{n+1} - y_n\}$ is in M. If $y'_n = y_{n+1} - x_n$, then

$$py'_{n+1} - y'_n = py_{n+2} - px_{n+1} - y_{n+1} + x_n = x_n .$$

Thus $p\{x_n\}$ in M implies $\{x_n\}$ in M, which proves (*b*).

To prove (*c*) we define an elementary periodic sequence $\bar{x} = \{\bar{x}_i\}$ by $\bar{x}_{nk} = 1$ for $n = 1, 2, \cdots$, and all other $x_i = 0$. If

$$y_{nk+i} = \frac{p^{k-i}}{p^k - 1} \qquad (i = 1, 2, \cdots, k; n = 0, 1, 2, \cdots),$$

then $py_{i+1} - y_i = \bar{x}_i$. Since $\lim_{n\to\infty} y_{nk} = (p^k - 1)^{-1}$ and this is not an integer (if $p > 2$, or if $p = 2$ and $k > 1$), $\bar{x}$ is not in M. The first multiple of $\bar{x}$ in M is $(p^k - 1)\bar{x}$. Therefore $\bar{x}$ represents an element of order $p^k - 1$ in $S - M$. Define $\bar{x}'$ by $\bar{x}'_i = \bar{x}_{i+1}$. Then as shown in the proof of (*a*), $p\bar{x}' = \bar{x}$ in $S - M$. Since $\bar{x}$ is of order $p^k - 1$, this can also be written $p^{k-1}\bar{x} = \bar{x}'$ in $S - M$. Similarly $p^{k-1}\bar{x}' = \bar{x}''$ where $\bar{x}''_i = \bar{x}'_{i+1}$. Since an arbitrary sequence of period k is a sum of the elementary periodic sequences $\bar{x}$, $\bar{x}'$, $\bar{x}''$, etc., and each of these is a multiple of $\bar{x}$, it follows that $\bar{x}$ generates a cyclic group of period $p^k - 1$ containing every sequence of period k. This proves (*c*).

Since $p^k \equiv 1 \pmod q$ has a solution k, (*d*) follows from (*c*).[19]

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[19] More information concerning sequences $\{x_n\}$ in M may be had by considering the corresponding power series $X(z) = \sum_1^\infty x_n z^n$. Then $X(z)$ is in M if there exists a power series $Y(z) = \sum_1^\infty y_n z^{n-1}$ such that $X(z) + py_1 = (p - z)Y(z)$. If $X(z)$ is in M, then formal differentiation shows that $X'(z) - Y(z) = (p - z)Y'(z)$ is in M. If $p = 2$, this can be used to show that, if x_n = a polynomial in n, then $\{x_n\}$ is in M.

VII. Applications

The results we have obtained enable us to extend a theorem of Eilenberg.[20] We shall first restate a special case of his theorem. Herein S^n is an n-sphere, $\pi_q(S^n)$ is the q^{th} homotopy group of S^n, K is an infinite complex, $H_q(K, \mathfrak{G})$ is the q^{th} cohomology group of K based on infinite cocycles with coefficients in $\mathfrak{G}$.

THEOREM OF EILENBERG. *If $H_q(K, \pi_q(S^n)) = H_{q+1}(K, \pi_q(S^n)) = 0$ for $q = n + 1, n + 2, \cdots$, then the homotopy classes of maps of K in S^n are in $1 - 1$ correspondence with the elements of the group $H_n(K, \pi_n(S^n))$.*

In case K is an m-dimensional manifold, then, by the $1 - 1$ correspondence between cells and their duals, interchanging boundary and coboundary, $H_q(K, \mathfrak{G})$ is isomorphic to the $(m - q)^{\text{th}}$ homology group $H^{m-q}(K, \mathfrak{G})$ again based on infinite cycles. We can therefore restate the above theorem in terms of homology. If, in addition, K is the complementary domain $S^m - X$ of a closed set X in an m-sphere S^m, we have by Theorem 4 that $H^{m-q}(K, \mathfrak{G})$ and $H^{m-q}(X, \mathfrak{G})$ are isomorphic. We may therefore restate Eilenberg's result as follows.

THEOREM 13. *If X is a closed subset of S^m and $H^q(X, \pi_{m-q}(S^n)) = H^{q-1}(X, \pi_{m-q}(S^n)) = 0$ for $q = 1, \cdots, m - n - 1$, then the homotopy classes of maps of $S^m - X$ in S^n are in $1 - 1$ correspondence with the elements of the group $H^{m-n}(X, \pi_n(S^n))$.*

COROLLARY. *If the assumptions of Theorem 13 are fulfilled, then the number of homotopy classes of maps of $S^m - X$ in S^n is a topological invariant of X.*

By choosing $n = m - 1$ in Theorem 13, we obtain

THEOREM 14. *If X is a closed subset of S^m then the homotopy classes of maps of $S^m - X$ into S^{m-1} are in a $1 - 1$ correspondence with the elements of $H^1(X, \pi_n(S^n))$.*

Since $\pi_n(S^n)$ is the infinite cyclic group, we have by Theorem 11:

COROLLARY. *If X is a solenoid in S^m then the set of homotopy classes of maps of $S^m - X$ into S^{m-1} has the power of the continuum.*

Finally Theorems 5, 9, and 13 give

THEOREM 15. *If the closed set $X \subset S^m$ is locally connected in the sense of homology in the first $m - n$ dimensions and the Vietoris homology groups $V^q(X, \pi_{m-q-1}(S^n)) = V^{q-1}(X, \pi_{m-q-1}(S^n)) = 0$ for $q = 0, 1, \cdots, m - n - 2$, then the homotopy classes of maps of $S^m - X$ in S^n are in $1 - 1$ correspondence with the elements of $V^{m-n-1}(X, \pi_n(S^n))$.*

VIII. Limits of bounding cycles

Since it has been customary to define the homology groups of infinite cycles of an infinite complex by regarding a cycle as ~ 0 whenever it is a limit of bounding cycles,[21] it is of interest to determine under what circumstances the group of bounding cycles is already closed.

[20] See footnote 5, Classification Theorem II p. 243, also p. 251. In the theorems of this section S^n $(n > 1)$ can be replaced by any arcwise connected space Y such that $\pi_i(Y) = 0$ for $i < n$. It should be kept in mind that $\pi_n(S^n)$ is isomorphic to the group of integers.

[21] See footnote 12, p. 691. Also, footnote 10, p. 229.

THEOREM 16. *If K is an infinite complex and the coefficient group $\mathfrak{G}$ has the division closure property*[22], *then an infinite cycle of K is expressible as a sum of finite bounding cycles if and only if it is a limit of bounding cycles.*

Let $\{K_i\}$ be a decreasing sequence of closed subcomplexes of K such that $K - K_i$ is finite, and $\prod K_i = 0$. Let C^q be a limit of bounding cycles, and C_i^q the part of C^q in $K - K_i$. Then, since C_i^q is a limit of bounding cycles, $K - K_i$ is finite, and $\mathfrak{G}$ has the division closure property, there is a chain C_i^{q+1} in $K - K_i$ such that $FC_i^{q+1} = C_i^q + D_i^q$ where $D_i^q \subset K_i$. Then

$$C^q = (C_1^q + D_1^q) + \sum_{i=2}^{\infty} \{C_i^q - C_{i-1}^q + D_i^q - D_{i-1}^q\}$$

is a sum of the indicated finite bounding cycles. The converse is trivial.

As an immediate consequence we have

THEOREM 17. *If every finite cycle of K bounds a chain (finite or infinite), and B^q is the group of infinite bounding cycles, and $\bar{B}^q$ its closure, then $\bar{B}^q - B^q$ is isomorphic to the homology group $\tilde{H}^q(K)$ of those cycles which are sums of finite cycles.*

By the corollary to Theorem 4, we have

COROLLARY. *If X is a closed subset of the n-sphere S^n and $\tilde{H}^q(X) = 0$, then in $K = S^n - X$ every limit of bounding cycles is bounding.*

It seems likely that this is still true if S^n is replaced by any finite n-manifold. However we can only prove a weaker result.

THEOREM 18. *If M^n is a finite n-manifold, X an absolute neighborhood retract in M^n, and $\mathfrak{G}$ has the division closure property, then every limit of bounding cycles of $K = M^n - X$ is bounding.*

Let $\{K_i\}$ be a decreasing sequence of closed polyhedral neighborhoods of X whose intersection is X. If the cycle C^q of K is a limit of bounding cycles, then by the proof of Theorem 16, C^q is homologous to a cycle Z_i in K_i for each i. Choose i so large that K_i can be retracted by a map g into X, and so that $g(K_i)$ is homotopic to the identity. This homotopy of Z_i into $g(Z_i)$ sweeps out a singular $(q + 1)$-chain which by subdivision and deformation reduces to a chain of $K_i - X$ having Z_i for boundary.

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[22] See footnote 12, p. 676. This property requires that for each integer m the subgroup of elements of $\mathfrak{G}$ divisible by m be closed. It is a sufficient condition for the group of bounding cycles of a finite complex to be closed.

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LOCAL UNIFORMIZATION ON ALGEBRAIC VARIETIES

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CONTENTS

A. Introduction

I. Uniformization of zero-dimensional valuations

1. In [10] (p. 650) we have proved a uniformization theorem for zero-dimensional valuations on an algebraic surface, over an algebraically closed ground field K (of characteristic zero). In the present paper we generalize this theorem to algebraic varieties, and on the basis of this generalization we obtain a solution of the problem of local uniformization in the classical case (i.e. when K is the field of complex numbers). The exact formulation of the generalized theorem, in its strongest form, will be given in A III and A IV. However, to begin with, we state here the following theorem which is literally a repetition of our theorem for surfaces, with the surface replaced by a variety, and which will be included in our final result:

Theorem U_1. *The Uniformization Theorem in invariantive form: Given a field Σ of algebraic functions of r independent variables, over an algebraically closed ground field K of characteristic zero, and given a zero-dimensional valuation B of Σ, there exists a projective model V of Σ on which the center of B is at a simple point P.*

This theorem is in effect entirely invariantive in nature: it refers exclusively to the field Σ and to the valuation B of Σ. It asserts the existence of *uni-*

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formizing parameters for B: they are the uniformizing parameters $t_1, t_2, \cdots, t_r$ at the simple point P (see [9], p. 258, [11], p. 199). Actually, however, the problem of local uniformization concerns not just the field Σ, but a specific projective model V_0 of Σ, given in advance. We must examine the situation more closely.

2. Let $(\xi_1, \cdots, \xi_n)$ be the general point of V_0 (in the affine space), and let P_0 be the center of B on V_0. The valuation B is represented on V_0 by a "branch" of some sort, and $t_1, \cdots, t_r$ are uniformizing parameters "along the branch." Let $(\eta_1, \cdots, \eta_m)$ be the general point of V. We assume that both P and P_0 are points at finite distance on the respective varieties V and V_0. The (non-homogeneous) coördinates η_i are expressible by means of formal power series of $t_1, \cdots, t_r$ ([9], p. 254). If K is the field of complex numbers, these power series (holomorphic functions of $t_1, \cdots, t_r$) will converge in some neighborhood of the initial values $t_1 = \cdots = t_r = 0$ and will yield (on V) a complete neighborhood E of the simple point P. However, the coördinates ξ_i, which are *rational* functions of $\eta_1, \cdots, \eta_m$, will be, in general, only *meromorphic* functions of $t_1, \cdots, t_r$ (i.e. quotients of power series). Following Walker [8], we shall say that the map of E on V_0 is *a wedge W on V_0* if $\xi_1, \cdots, \xi_n$ are holomorphic functions of $t_1, \cdots, t_r$.[1] What we want is a set of wedges on V_0 and—ultimately—a proof that a complete neighborhood of P_0—and hence also V_0 itself (since V_0 is bicompact)—can be covered by a finite set of wedges.[2] Hence the knowledge that there exist uniformizing parameters for B is not sufficient. We ought to be able to exhibit parameters $t_1, \cdots, t_r$ such that the ξ_i be holomorphic functions of the t's. Now let us observe that any polynomial ψ in $\eta_1, \cdots, \eta_m$ is an holomorphic function of $t_1, \cdots, t_r$ and that ψ vanishes for $t_1 = \cdots = t_r = 0$ if and only if ψ vanishes at P. Hence every element of the *quotient ring* $Q(P)$ of the point P is an holomorphic function of $t_1, \cdots, t_r$. Therefore, in order that the ξ's be holomorphic functions of the t's, it is sufficient that they belong to the quotient ring $Q(P)$. If that is the case, then not only the ξ's, but also *the entire quotient ring $Q(P_0)$ will belong to $Q(P)$*. Namely, since P_0 is the center of B on V_0, any polynomial φ in $\xi_1, \cdots, \xi_n$ which does not vanish at P_0 has value zero in B, and therefore φ, as an element of $Q(P)$, must be a unit.

The above considerations suggest the following stronger formulation of the uniformization theorem:

THEOREM U_2. (*The uniformization theorem in projective form*): *Given a projective model V_0 of Σ and given a zero-dimensional valuation B of Σ, with center P_0 on V_0, there exists a projective model V of Σ on which the origin of B is at a simple*

[1] Note that the mapping is then single-valued on E but that the inverse is not necessarily single-valued on W, whence W need not be an open set. Note also that P is mapped upon P_0.

[2] A point Q_0 of V_0 is said to be *covered by the wedge* W, if Q_0 is the image of a point Q of the open set E.

point P and which is such that the quotient ring of P_0 is a subring of the quotient ring of $P(Q(P_0) \subsetneqq Q(P))$.

3. So far we have restricted ourselves to: (a) valuations of dimension zero; (b) ground fields which are algebraically closed. An analysis of the theorem U_2 (see A III) will show, however, that both restrictions are quite illusory, in the sense that the proof of Theorem U_2 leads automatically to the necessity of proving an analogous theorem under the most general hypotheses: (a) B is a valuation of any dimension; (b) K is an arbitrary field (of characteristic zero). But before we undertake this analysis, we proceed to show that from Theorem U_2 follows almost immediately the solution of the problem of local uniformization in the classical case.

II. Solution of the classical problem of local uniformization

4. The ground field K being quite arbitrary, any zero-dimensional valuation B of the field[3] Σ defines an homomorphic mapping of Σ upon (K', ∞), where K' is an algebraic extension of K (finite or infinite). The valuation-theoretic value of an element η of Σ shall be denoted by $v(\eta)$; here $v(\eta)$ is an element of an ordered abelian (additive) group Γ, *the value group of B*. The element c of K' upon which η is mapped, or the symbol ∞ if η is mapped on ∞, shall be called the *function-theoretic value of η*, or also the *B-residue of η*, and shall be denoted by $B(\eta)$. The field K' itself is the *residue field* of B. The relationship between $v(\eta)$ and $B(\eta)$ is the following: (a) $B(\eta) = \infty$, if and only if $v(\eta) < 0$; (b) $B(\eta) = 0$, if and only if $v(\eta) > 0$; (c) if $B(\eta) = \vartheta$ and if $f(\vartheta) = 0$ is the irreducible equation for ϑ, with coefficients in K, then $v(f(\eta)) > 0$. The elements η for which $v(\eta) \geqq 0$ form a ring $\mathfrak{B}$, called the *valuation ring* of B. By (b) and (c), the mapping $\eta \rightarrow B(\eta)$ is an homomorphic mapping of $\mathfrak{B}$ upon K'. The elements η of $\mathfrak{B}$ such that $B(\eta) = 0$ form a prime maximal ideal $\mathfrak{P}$ in $\mathfrak{B}$, and we have $\mathfrak{B} \sim \mathfrak{B}/\mathfrak{P} \cong \mathsf{K}'$.

Following Dedekind, we define *a place $\mathfrak{P}$ of the field Σ* as an homomorphic mapping of Σ upon (K', ∞), where K' is an algebraic extension of K; provided $c \rightarrow c$, for any c in K. We have just shown that any zero-dimensional valuation B of Σ defines a place. *Conversely, given a place $\mathfrak{P}$, it determines uniquely a zero-dimensional valuation of Σ.* Namely, the elements of Σ which are mapped upon elements of K', form a ring $\mathfrak{B}$ with the following property: if $\eta \not\subset \mathfrak{B}$ then $1/\eta \subset \mathfrak{B}$. Hence *$\mathfrak{B}$ is a valuation ring* ([3], p. 110), and moreover $\mathsf{K} \subset \mathfrak{B}$. The elements of $\mathfrak{B}$ which are mapped by $\mathfrak{P}$ upon the zero element of K', form the prime ideal of non-units in $\mathfrak{B}$. We denote this ideal by the same letter $\mathfrak{P}$. Since $\mathfrak{B}/\mathfrak{P} \cong \mathsf{K}'$ and since K' is an algebraic extension of K, it follows that the valuation B determined by the ring $\mathfrak{B}$ is zero-dimensional, with K' as residue field. This proves the assertion.

Thus the two terms: "*zero-dimensional valuation*" and "*place*," are inter-

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[3] We tacitly assume that only such valuations of Σ are considered in which the elements c of K $(c \neq 0)$ have value zero.

changeable. We speak of a valuation if we wish to focus our attention on the mapping $\eta \to v(\eta) \subset \Gamma$; we speak of a place if we wish instead to emphasize the mapping $\eta \to B(\eta) \subset (\mathsf{K}', \infty)$. In this second case we shall also use the symbol $\mathfrak{P}(\eta)$ to denote the function-theoretic value of η.

5. DEFINITION. The totality of places of Σ is called the *Riemann manifold* of Σ. This manifold shall be denoted by $\boldsymbol{M}$.

If K is algebraically closed, then $\mathsf{K}' = \mathsf{K}$ for every place of Σ. Let now K be the field of complex numbers. Then (K, ∞) is a sphere H, the sphere of the complex variable. $\boldsymbol{M}$ becomes an Hausdorff space if we define *open sets* in $\boldsymbol{M}$ in the following (well-known) fashion:

Let $\eta_1, \cdots, \eta_m$ be any finite set of elements of Σ and let $E_1, E_2, \cdots, E_m$ be open sets in H. *The totality of all places $\mathfrak{P}$, or zero-dimensional valuations B, such that $B(\eta_i) \subset E_i$, is an open set in $\boldsymbol{M}$.* The open sets of $\boldsymbol{M}$ just defined shall form a *basis* for the totality of open sets on $\boldsymbol{M}$. It is obvious that this definition is equivalent with the following: *if $(\eta_1, \cdots, \eta_m)$ is the general point of a projective model V of Σ, then the zero-dimensional valuations B whose centers lie in a given open subset of V form an open set in $\boldsymbol{M}$.*

Given a projective model V_0 of Σ, every "point" $\mathfrak{P}$ of $\boldsymbol{M}$ lies over a unique point P_0 of V_0 : here P_0 is the center of the zero-dimensional valuation B determined by the place $\mathfrak{P}$. The mapping $\mathfrak{P} \to P_0$ is obviously a continuous mapping of $\boldsymbol{M}$ on V_0, since any open set on V_0 is the map of an open set on $\boldsymbol{M}$.

By Theorem U_2 it follows that *$\boldsymbol{M}$ can be covered by a collection of open sets whose images on V_0 are wedges $W_1, W_2, \cdots$.* We shall denote these open sets by $W_1', W_2', \cdots$. In the notation of Theorem U_2, each W' is the totality of all places which lie over points of a sufficiently small neighborhood E of the simple point P (so small as to insure the convergence of the power series which give the coördinates of the general point of V and of V_0 in terms of $t_1, \cdots, t_r$). By a preceding definition (see footnote[2]) it follows that if a set W' contains a point $\mathfrak{P}$ of $\boldsymbol{M}$, and if $\mathfrak{P}$ lies over the point A_0 of V_0, then the wedge W "covers" A_0. Namely, let A be the point of E which corresponds to the place $\mathfrak{P}$. Since W is a single-valued image of E, the image of A must be a point A_0^* of W such that *every* place which lies over A lies over A_0^*.[4] But the place $\mathfrak{P}$ lies over A and over A_0. Hence $A_0 = A_0^*$, and consequently W covers A_0.

We conclude that in order to show that V_0 can be covered by a *finite number* of wedges W_i, it is sufficient to show that the Riemann manifold $\boldsymbol{M}$ can be covered by a finite number of open sets W_i', and for that it is sufficient to show (in view of Theorem U_2) that *$\boldsymbol{M}$ is a bicompact space.* Our original proof of the bicompactness of $\boldsymbol{M}$ was tolerably complicated. We are, however, indebted to Chevalley and Tukey for the remark that the bicompactness of $\boldsymbol{M}$ follows

[4] Since $Q(P_0) \subsetneqq Q(P)$, the ξ's are rational functions g of $\eta_1, \cdots, \eta_m$ in which the denominators do not vanish at P. Hence if E is a sufficiently small neighborhood of P, these denominators will be $\neq 0$ in E. For any place $\mathfrak{P}$ lying over A, the function-theoretic values of the ξ's will then be equal to the values of the rational functions g at A.

immediately from the theorem of Tychonoff ([5], [6]) on the bicompactness of direct products of bicompact spaces. To see this, we consider the Hausdorff space Ω of *all* mappings of Σ upon H.[5] By Tychonoff's theorem, Ω is bicompact, since H is bicompact.

The Riemann manifold $\boldsymbol{M}$ of Σ is a subset of Ω, consisting of the *homomorphic* mappings upon H (i.e. upon (K, ∞)). It is a straightforward matter to show that *$\boldsymbol{M}$ is a closed subset of* Ω.[6] Hence $\boldsymbol{M}$ is bicompact. This settles the problem of local uniformization for algebraic varieties.

6. The following result, though purely negative, is nevertheless of interest: *if $r > 1$, then the Riemann manifold $\boldsymbol{M}$ of Σ is not metrizable.* To show this it is only necessary to show that the open subsets of $\boldsymbol{M}$ do not possess a countable basis (Urysohn's metrization theorem; see, for instance, [1], p. 88). Let us suppose that such a basis exists. Then, given any basis (countable or not) for the open sets on $\boldsymbol{M}$, that basis will necessarily contain a countable subset also forming a basis ([1], p. 78). It follows that there exists a countable basis $\{N_1, N_2, \cdots\}$ consisting of open sets defined in the preceding section. Each set N_i is defined by a finite set of elements in Σ, say $\eta_{i1}, \eta_{i2}, \cdots, \eta_{i\nu_i}$, and by a corresponding set of open sets on H, say $E_{i1}, E_{i2}, \cdots, E_{i\nu_i}$. Altogether we have a countable set of elements η_{ij} in Σ. We assert that this set enjoys the following property: *there cannot exist two distinct places $\mathfrak{P}_1$, $\mathfrak{P}_2$ such that the function-theoretic value of each η_{ij} at $\mathfrak{P}_1$ is the same as the function-theoretic value of the same element η_{ij} at $\mathfrak{P}_2$.* In other words: *the function-theoretic values of the elements η_{ij} at a place $\mathfrak{P}$ determine the place $\mathfrak{P}$ uniquely.* The proof is immediate. Let us suppose that there exist two distinct places $\mathfrak{P}_1$, $\mathfrak{P}_2$ such that $\mathfrak{P}_1(\eta_{ij}) = \mathfrak{P}_2(\eta_{ij})$, $i = 1, 2, \cdots; j = 1, 2, \cdots, \nu_i$. If a set N_i contains $\mathfrak{P}_1$, we must have $E_{ij} \supset \mathfrak{P}_1(\eta_{ij})$, $j = 1, 2, \cdots, \nu_i$, and since $\mathfrak{P}_1(\eta_{ij}) = \mathfrak{P}_2(\eta_{ij})$, that implies that N_i also contains $\mathfrak{P}_2$. This is in contradiction with the hypothesis that $\{N_1, N_2, \cdots\}$ is a basis for the open sets of the *Hausdorff* space $\boldsymbol{M}$.

We complete our proof that $\boldsymbol{M}$ is not metrizable by showing that *if $r > 1$, then given any countable set of elements in Σ, say $\eta_1, \eta_2, \cdots, \eta_n, \cdots$, there always exist two distinct places $\mathfrak{P}_1$, $\mathfrak{P}_2$ such that $\mathfrak{P}_1(\eta_i) = \mathfrak{P}_2(\eta_i)$, for all i.* For the proof, we consider some *normal* projective model V of Σ in an affine space S_n ([9], p. 279). Let $(\xi_1, \cdots, \xi_n)$ be the general point of V. Since V is normal,

[5] It should be understood that open sets in Ω are defined in the same fashion as the open sets of the Riemann manifold were defined above.

[6] Let $\overline{\boldsymbol{M}}$ be the closure of $\boldsymbol{M}$ in Ω and let f_0 be an element of $\overline{\boldsymbol{M}}$. We have to show that f_0 is a homomorphic mapping of Σ on (K, ∞). Let ω_1, ω_2 be any two elements of Σ and let $\omega_3 = \omega_1 + \omega_2$. Let us assume that $f_0(\omega_i) \neq \infty$ $(i = 1, 2, 3)$. Consider the open set N in Ω which consists of all mapping f such that $| f(\omega_i) - f_0(\omega_i) | < \epsilon/3$, $i = 1, 2, 3$, ϵ—an arbitrary positive real number. Since $f_0 \subset \overline{\boldsymbol{M}}$, N contains an element φ which is an homomorphic mapping of Σ on (K, ∞). We will have then $| \varphi(\omega_i) - f_0(\omega_i) | < \epsilon/3$, and also $\varphi(\omega_3) = \varphi(\omega_1) + \varphi(\omega_2)$. Hence $|f_0(\omega_3) - f_0(\omega_1) - f_0(\omega_2)| < \epsilon$, and since ϵ is arbitrary, we conclude that $f_0(\omega_3) = f_0(\omega_1) + f_0(\omega_2)$. In a similar fashion it follows that $f_0(\omega_1 \cdot \omega_2) = f_0(\omega_1) f_0(\omega_2)$. The case in which some of the $f_0(\omega_i)$ is ∞ is treated in a similar fashion.

the ring $\mathfrak{o} = K[\xi_1, \cdots, \xi_n]$ is integrally closed in Σ. In the sense of *quasi-gleicheit* of van der Waerden we have $\eta_i = \mathfrak{A}_i/\mathfrak{B}_i$, where $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are power products of minimal ($(r-1)$-dimensional) prime ideals of $\mathfrak{o}$. The prime factors of $\mathfrak{A}_i$ are distinct from those of $\mathfrak{B}_i$. Hence the ideal $(\mathfrak{A}_i, \mathfrak{B}_i)$ defines on V a subvariety W_i of dimension $\leqq r - 2$. The countable set $\{W_1, W_2, \cdots\}$ cannot fill the entire variety V, which is of dimension r. Hence there exists on V a point $P(a_1, \cdots, a_n)$ which is on none of the varieties W_i. For a given i, the prime ideal $\mathfrak{p} = (\xi_1 - a_1, \cdots, \xi_n - a_n)$ in $\mathfrak{o}$ may divide one of the ideals $\mathfrak{A}_i, \mathfrak{B}_i$, but cannot divide both. From this it follows that η_i can be expressed in the form: $\eta_i = \varphi_i(\xi_1, \cdots, \xi_n)/\psi_i(\xi_1, \cdots, \xi_n)$, where φ_i and ψ_i are polynomials and where $\varphi_i(a_1, \cdots, a_n)$, $\psi_i(a_1, \cdots, a_n)$ are not both zero.[7] Now consider any place $\mathfrak{P}$ of the field Σ which lies over the point P. The function-theoretic value of η_i at $\mathfrak{P}$ is then uniquely determined by P. Namely, if $\psi_i(a_1, \cdots, a_n) \neq 0$, then $\mathfrak{P}(\eta_i) = \varphi_i(a_1, \cdots, a_n)/\psi_i(a_1, \cdots, a_n)$. If $\psi_i(a_1, \cdots, a_n) = 0$, then $\mathfrak{P}(\eta_i) = \infty$. *Since, for $r > 1$, there are infinitely many places $\mathfrak{P}$ which lie over P* (for instance, all the algebraic branches through P), our assertion is proved.

III. The general uniformization theorem

7. Let us assume for a moment that Theorem U_2 has been established, and let us consider a valuation B of Σ, *of an arbitrary dimension $s(s < r)$*. Given a projective model V of Σ, we define the *center of B on V* as follows. Let $(\xi_1, \cdots, \xi_n)$ be the non-homogeneous coördinates of the general point of V. Without loss of generality we may assume that the elements ξ_i have non-negative values $v(\xi_i)$ in B. Then the entire ring $\mathfrak{o} = K[\xi_1, \cdots, \xi_n]$ will be contained in the valuation ring $\mathfrak{B}$ of B. Let $\mathfrak{P}$ be the prime ideal of non-units in $\mathfrak{B}$ and let $\mathfrak{p} = \mathfrak{P} \cap \mathfrak{o}$. The ideal $\mathfrak{p}$ is prime and shall be referred to as *the prime ideal of $\mathfrak{B}$ in $\mathfrak{o}$*; $\mathfrak{p}$ consists of those elements ω of $\mathfrak{o}$ for which $v(\omega) > 0$. Let W be the irreducible algebraic subvariety of V defined by the prime ideal $\mathfrak{p}$. Since $\mathfrak{B}/\mathfrak{P}$ is of degree of transcendency s over K and since $\mathfrak{o}/\mathfrak{p}$ is a subring of $\mathfrak{B}/\mathfrak{P}$, it follows that W is of dimension $\leqq s$. This subvariety W is called *the center of B on V*.

The residue field $\mathfrak{B}/\mathfrak{P}$ of B is a field Σ', of degree of transcendency s over K. We consider an arbitrary zero-dimensional valuation B' of Σ'/K. Compounding B with B' we get a zero-dimensional valuation B_0 of Σ. Let V be a projective model of Σ on which the center of B_0 is a simple point P. Let W be the center of B on V. Since B_0 is compounded with B, *the center of B_0 lies on the center of B*,[8] i.e. P is a point of W. Hence W, containing a simple point of V, *must*

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[7] The assertion is obvious. If, for instance, $\mathfrak{B}_i \not\equiv 0(\mathfrak{p})$, then we take an element $\psi_i(\xi_1, \cdots, \xi_n)$ in $\mathfrak{o}$ such that $\psi_i \equiv 0(\mathfrak{B}_i)$, $\psi_i \not\equiv 0(\mathfrak{p})$. Then $\psi_i \cdot \eta_i$ is in $\mathfrak{o}$, $\psi_i \eta_i = \varphi_i(\xi_1, \cdots, \xi_n)$.

[8] This follows from the fact that any element of Σ whose value in B is positive has also positive value in B_0.

itself be a simple subvariety of V ([11], p. 221). *Thus we have a projective model V on which the center of B is a simple subvariety W* (of dimension $\leqq s$).

This result is a generalization of Theorem U_1, but actually it is implied by this theorem. However, in our proof, the relationship between the uniformization of B and that of the compounded valuation B_0 will be inverted: *we first uniformize B and we then use this result in order to uniformize B_0*. The reason for this is that the *value group of B is of lower rank than that of B_0* ([3], p. 113) and that our proof proceeds *by induction with respect to the rank.* From this point of view it begins to appear clearly that the center of gravity of our proof will be found, not in the uniformization of valuations *of dimension zero* and *of arbitrary rank,* but rather in the uniformization of valuations *of rank* 1 *and of arbitrary dimension.* The rest of the proof will be a relatively simple induction from rank σ to rank $\sigma + 1$.

8. The above program calls for another reduction and for a simultaneous generalization of the terms of the problem. Let us adjoin to the ground field K s elements $\zeta_1, \cdots, \zeta_s$ of Σ which are mapped upon algebraically independent elements of the residue field Σ' of B. Every element ($\neq 0$) of the field $\mathsf{K}^* = \mathsf{K}(\zeta_1, \cdots, \zeta_s)$ has value zero in B, and it is therefore permissible to take K^* as new ground field for Σ. *With respect to this new ground field, our valuation B is of dimension zero.* This is the reduction. However, K^* is not an algebraically closed field. It thus appears that the uniformization of valuations of an arbitrary dimension, in the case of an algebraically closed ground field, is in essence nothing else than the uniformization of zero-dimensional valuation in the case of an arbitrary ground field.

These considerations, *and the additional requirement concerning the quotient rings* (Theorem U_2) lead us to state our *general uniformization theorem* as follows:

THEOREM U_3. *Let Σ be a field of algebraic functions of r independent variables, over an arbitrary ground field K of characteristic zero, and let B be an s-dimensional valuation of Σ. Let, moreover, V be a projective model of Σ and let W be the center of B on V. There exists a projective model V' of Σ, on which the center of B is a simple subvariety W' and such that the quotient ring of W is a subring of the quotient ring of W'.*[9]

IV. Reduction to an hypersurface. The main theorem

9. Let us suppose that Theorem U_3 has already been proved in the case when the given projective model V is an hypersurface, i.e. an r-dimensional variety V lying in an $(r + 1)$-dimensional projective space. We propose to show that Theorem U_3 then follows quite generally.

Let $(\xi_1, \cdots, \xi_n)$ be the general point of V in the affine space S_n and let $\mathfrak{o} = \mathsf{K}[\xi_1, \cdots, \xi_n]$. We choose in $\mathfrak{o}$ r elements $\eta_1, \cdots, \eta_r$ such that $\mathfrak{o}$ is

[9] Note that from $Q(W) \subseteqq Q(W')$ follows incidentally that *dimension $W \leqq$ dimension W'*.

integrally dependent on $\mathsf{K}[\eta_1, \cdots, \eta_r]$.[10] Then we fix in $\mathfrak{o}$ a primitive element η_{r+1} of the field $\Sigma/\mathsf{K}(\eta_1, \cdots, \eta_r)$. Let $f(\eta_1, \cdots, \eta_r, \eta_{r+1}) = 0$ be the irreducible equation, over K, between the $r + 1$ elements $\eta_1, \cdots, \eta_{r+1}$, and let V^* be the hypersurface defined by this equation. Let $\mathfrak{o}^* = \mathsf{K}[\eta_1, \cdots, \eta_r, \eta_{r+1}]$ and let W and W^* be the centers of the given s-dimensional valuation B on V and V^* respectively. Since $\mathfrak{o}$ is integrally dependent on $\mathfrak{o}^*$, and since the prime $\mathfrak{o}^*$-ideal determined by W^* is the contracted ideal of the prime $\mathfrak{o}$-ideal for W, it follows that dimension W^* = dimension W.

By our assumption, there exists a projective model V' of Σ on which the center of B is a simple subvariety W' and such that $Q(W')$ (quotient ring of W') contains $Q(W^*)$. Now the quotient ring of a simple subvariety is integrally closed in Σ ([11], p. 220). Hence $Q(W')$, being integrally closed, *is the intersection of all the valuation rings which contain* $Q(W')$ ([3], p. 111). Since $Q(W') \supseteqq Q(W^*)$, every valuation ring containing $Q(W')$ also contains $Q(W^*)$. Since $\mathfrak{o}$ is integrally dependent on $Q(W^*)$, every valuation ring containing $Q(W^*)$ also contains $\mathfrak{o}$. *Hence* $\mathfrak{o} \subseteq Q(W')$. *Let* $\mathfrak{p}$ *be the prime ideal of* W *in* $\mathfrak{o}$ (i.e. the prime ideal of B in $\mathfrak{o}$). Every element η of $\mathfrak{o}$ which is not in $\mathfrak{p}$ is such that $v(\eta) = 0$ in B. Such an element η, as an element of $Q(W')$, must be a unit, since also W' is the center of B, on V'. This shows that $\mathfrak{o}_\mathfrak{p}$, i.e. $Q(W)$, is contained in $Q(W')$, which proves Theorem U_3.

10. Our main theorem, to the proof of which most of the paper is devoted, concerns hypersurfaces. This theorem asserts, in part, much more than Theorem U_3, and namely it asserts that *for hypersurfaces, the uniformization of any given valuation of rank* 1 *and dimension zero can be effected by Cremona transformations of the ambient space.* This result is naturally of added interest. We believe that the second part of the paper (Part B), where we build up our Cremona transformations, contains the necessary material for a proof that also valuations of rank > 1 can be uniformized by Cremona transformations. However, we do not wish to overload the present paper with considerations which are not absolutely essential for the proof of the general uniformization theorem. On the other hand, our proof of this last theorem (see C IV) is absolutely dependent on the knowledge that valuation of rank 1 can be uniformized by Cremona transformations (of the type described below in the main theorem).

We also wish to point out that from the main theorem it follows directly that a rank 1 valuation B of arbitrary dimension s can also be uniformized by a Cremona transformation, *provided the center of* B *on the hypersurface* $f = 0$ *is a subvariety of dimension exactly* s. Namely, with this proviso, s of the elements $x_1, \cdots, x_{r+1}$ will have algebraically independent B-residues. If say $B(x_1), \cdots, B(x_s)$ are algebraically independent over K, we adjoin $x_1, \cdots, x_s$ to the ground field K. We take $\mathsf{K}(x_1, \cdots, x_s)$ as new ground field and we apply our main theorem to the *zero-dimensional* valuation B of $\Sigma/\mathsf{K}(x_1, \cdots, x_s)$. However,

[10] By a "normalization" theorem due to Emmy Noether, it is possible to take for $\eta_1, \cdots, \eta_r$ linear forms in $\xi_1, \cdots, \xi_n$, with "non special" coefficients in K.

in this case, the expressions $\varphi_i(X_1, \cdots, X_{r+1})$, $i = 1, 2, \cdots, r + 1$, in the equations $x_i = \varphi_i(X_1, \cdots, X_{r+1})$ (given below), will be polynomials only in $X_{s+1}, \cdots, X_{r+1}$, but they will be in general *rational* functions in $X_1, \cdots, X_s$. (Except for $i = 1, 2, \cdots, s$, for then φ_i is simply X_i.)

MAIN THEOREM. *Let*

$$f(x_1, x_2, \cdots, x_{r+1}) = 0$$

be an irreducible hypersurface V in S_{r+1}, over an arbitrary ground field K (*of characteristic zero*), *and let B be a zero-dimensional valuation, of rank* 1, *of the field* Σ (*field of rational functions on V*), *such that the valuation ring* $\mathfrak{B}$ *of B contains the x_i* ($i = 1, 2, \cdots, r + 1$). *Let P be the center of B on V. There exists a Cremona transformation of the form*:

$$x_i = \varphi_i(X_1, X_2, \cdots, X_{r+1}), \qquad i = 1, 2, \cdots, r + 1,$$

the φ_i-polynomials, with coefficients in K, *such that the new elements X_i also belong to* $\mathfrak{B}$ *and such that on the transformed hypersurface*

$$F(X_1, X_2, \cdots, X_{r+1}) = 0,$$

the center of B is a simple point P'.

The restriction: $x_i \subset \mathfrak{B}$, implies no loss of generality, since we may always perform a preliminary projective transformation on the x_i so as to satisfy this condition. The condition $Q(P') \supseteqq Q(P)$ is here automatically satisfied, since the ring $\mathsf{K}[x_1, \cdots, x_{r+1}]$ is a subring of $\mathsf{K}[X_1, \cdots, X_{r+1}]$ (the φ_i being polynomials).

B. THE ALGORITHM OF PERRON AND THE ASSOCIATED CREMONA TRANSFORMATIONS

It is well known that given a *rank one* valuation B (i.e. a valuation whose value group consists of real numbers) of a field Σ of algebraic functions of r independent variables, the value group Γ of B is of *rational rank* $m \leqq r$ ([3], p. 116). There exist then in Γ m rationally independent real numbers, say $\tau_1, \cdots, \tau_m$, and every element of Γ is rationally dependent on $\tau_1, \cdots, \tau_m$. In the case of surfaces, m is at most 2. If $m = 2$, we have used in our paper [10] the algorithm of continued fractions for the ratio τ_1/τ_2. With each successive step of this algorithm (i.e. for each convergent fraction of τ_1/τ_2) we have associated a Cremona transformation in a field of two independent variables (see [10], p. 653, (8)). In the quoted paper we have assumed throughout that the ground field is algebraically closed.

Our present investigation depends on a two-fold generalization of the above procedure. First, we need an algorithm of simultaneous approximation of the set of real numbers $\tau_1, \cdots, \tau_m$. We use an algorithm due to O. Perron [7]; see also [2], Chapter IV. In the second place, given a set of elements $x_1, \cdots, x_m$, such that $v(x_i) = \tau_i$, we have to provide ourselves with a sequence of Cremona transformations in the field $\mathsf{K}(x_1, \cdots, x_m)$ running parallel to

Perron's algorithm. The main difficulties of this step arise when K is not algebraically closed, or better, when we are dealing with a zero-dimensional valuation whose residue field is a proper algebraic extension of K.

I. The algorithm of Perron

1. Let $\tau_1, \tau_2, \cdots, \tau_m$ be rationally independent positive real numbers. We consider the transformation (referred to in the sequel as the *elementary transformation* of Perron):

$$(1) \qquad \tau_1 = \tau_m^{(1)},\ \tau_2 = \tau_1^{(1)} + a_2^{(0)} \cdot \tau_m^{(1)},\ \cdots,\ \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)} \tau_m^{(1)},$$

where

$$(1') \qquad a_j^{(0)} = [\tau_j/\tau_1], \qquad j = 2, \cdots, m.$$

The real numbers $\tau_1^{(1)}, \cdots, \tau_m^{(1)}$ are again positive and rationally independent. We apply the same transformation to the numbers $\tau_i^{(1)}$, getting a new set of m positive, rationally independent numbers $\tau_1^{(2)}, \cdots, \tau_m^{(2)}$, and so we continue indefinitely. In this fashion we obtain, for each integer h, a set of m positive, rationally independent real numbers: $\tau_1^{(h)}, \cdots, \tau_m^{(h)}$, where:

$$(2) \quad \tau_1^{(h-1)} = \tau_m^{(h)},\ \tau_2^{(h-1)} = \tau_1^{(h)} + a_2^{(h-1)} \tau_m^{(h)},\ \cdots,\ \tau_m^{(h-1)} = \tau_{m-1}^{(h)} + a_m^{(h-1)} \tau_m^{(h)},$$

and

$$(2') \qquad a_j^{(h-1)} = [\tau_j^{(h-1)}/\tau_1^{(h-1)}].$$

The resulting transformation (referred to in the sequel as the *transformation of Perron*) from $\tau_1, \cdots, \tau_m$ to $\tau_1^{(h)}, \cdots, \tau_m^{(h)}$ is of the form:

$$(3) \qquad \tau_i = A_i^{(h)} \tau_1^{(h)} + A_i^{(h+1)} \tau_2^{(h)} + \cdots + A_i^{(h+m-1)} \tau_m^{(h)}, \qquad i = 1, 2, \cdots, m.$$

Here the coefficients $A_i^{(j)}$ are *non-negative integers*

$$(4) \qquad A_i^{(j)} \geqq 0,$$

and, moreover, the transformation (3) is *unimodular*:

$$(4') \qquad \begin{vmatrix} A_1^{(h)}, \cdots, A_1^{(h+m-1)} \\ \cdots\cdots\cdots\cdots\cdots \\ A_m^{(h)}, \cdots, A_m^{(h+m-1)} \end{vmatrix} = (-1)^{h(m-1)}.$$

(See [2], p. 437, 439, formulas (5), (14).)

Of fundamental importance is the following: *the ratios $A_i^{(h)}/A_1^{(h)}$ are convergent fractions for τ_i/τ_1* i.e. ([2], p. 440, (18)):

$$(5) \qquad \lim_{h\to\infty} \frac{A_i^{(h)}}{A_1^{(h)}} = \frac{\tau_i}{\tau_1}.$$

2. By means of the algorithm of Perron we derive a theorem concerning *rationally dependent* real numbers. Let $\tau_1, \cdots, \tau_m$ be, as before, rationally

independent positive real numbers and let τ_{m+1} be a positive real number which depends rationally on $\tau_1, \cdots, \tau_m$.

THEOREM 1. *There exists a unimodular transformation, with integral non-negative coefficients*:

$$\tau_i = \sum_{j=1}^{m+1} \beta_{ij}\tau_j^*, \qquad i = 1, 2, \cdots, m+1, \tag{6}$$

such that

$$\tau_{m+1}^* = 0, \qquad \tau_i^* > 0, \qquad i = 1, 2, \cdots, m. \tag{6'}$$

PROOF: Let

$$\lambda\tau_{m+1} = \lambda_1\tau_1 + \cdots + \lambda_m\tau_m, \qquad \lambda > 0, \tag{7}$$

be the relation of rational dependence between $\tau_1, \cdots, \tau_m, \tau_{m+1}$. Here $\lambda, \lambda_1, \cdots, \lambda_m$ are integers and $(\lambda, \lambda_1, \cdots, \lambda_m) = 1$. We apply to $\tau_1, \cdots, \tau_m$ the transformation of Perron (3). By substituting into (7) we get:

$$\lambda\tau_{m+1} = \lambda_1^{(h)}\tau_1^{(h)} + \cdots + \lambda_m^{(h)}\tau_m^{(h)}, \tag{8}$$

where

$$\lambda_i^{(h)} = \lambda_1 A_1^{(h+i-1)} + \lambda_2 A_2^{(h+i-1)} + \cdots + \lambda_m A_m^{(h+i-1)} \qquad i = 1, 2, \cdots, m. \tag{9}$$

We take h sufficiently high, so as to satisfy the condition $\lambda_i^{(h)} > 0$. This is possible, in view of (5), since $\lambda_1\tau_1 + \cdots + \lambda_m\tau_m > 0$.[11] Since the transformation (9) is unimodular, the relation $(\lambda, \lambda_1, \cdots, \lambda_m) = 1$ implies that also $(\lambda, \lambda_1^{(h)}, \cdots, \lambda_m^{(h)}) = 1$. Let, say, $\lambda_1^{(h)}$ not be divisible by λ, and let $\lambda_1^{(h)} = \lambda\mu + \lambda'$,

$$0 < \lambda' < \lambda. \tag{10}$$

We consider the transformation:

$$\tau_{m+1} = \tau_1' + \mu\tau_{m+1}', \quad \tau_1^{(h)} = \tau_{m+1}', \quad \tau_i^{(h)} = \tau_i', \qquad i = 2, \cdots, m. \tag{10'}$$

The $m+1$ real numbers τ_i' are positive, since $\lambda\tau_{m+1} > \lambda_1^{(h)}\tau_1^{(h)}$, by (8), whence $\lambda\tau_{m+1} > \lambda\mu\tau_1^{(h)}$, i.e. $\lambda\tau_1' > 0$. From (8) we find:

$$\lambda'\tau_{m+1}' = \lambda_1'\tau_1' + \cdots + \lambda_m'\tau_m', \tag{7'}$$

where $\lambda_1' = \lambda$, $\lambda_i' = -\lambda_i^{(h)}$, $i = 2, \cdots, m$. In view of (10) and (7) we have achieved a reduction for λ, and this by a transformation (from the τ_i to the τ_i') which is unimodular and non-negative (i.e. with non-negative integral coefficients). Ultimately we will get by a unimodular non-negative transformation a set of positive real numbers $\bar\tau_1, \bar\tau_2, \cdots, \bar\tau_{m+1}$, such that

$$\bar\tau_{m+1} = \bar\lambda_1\bar\tau_1 + \cdots + \bar\lambda_m\bar\tau_m. \tag{11}$$

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[11] Note also that, in view of (5), $A_1^{(h)} \neq 0$, when h is sufficiently high, whence, by (4), $A_1^{(h)} > 0$.

A Perron transformation (3) applied to the rationally independent numbers $\bar{\tau}_1, \cdots, \bar{\tau}_m$ will lead to $\bar{\tau}_1^{(h)}, \cdots, \bar{\tau}_m^{(h)}$ and to a new relation:

$$\bar{\tau}_{m+1} = \bar{\lambda}_1^{(h)} \bar{\tau}_1^{(h)} + \cdots + \bar{\lambda}_m^{(h)} \bar{\tau}_m^{(h)}, \tag{11'}$$

in which the coefficients $\bar{\lambda}_i^{(h)}$ are non-negative, provided h is sufficiently high (the argument is the same as that applied to the coefficients $\lambda_i^{(h)}$ in (9)). We then put:

$$\begin{cases} \bar{\tau}_i^{(h)} = \overset{*}{\tau}_i, & i = 1, 2, \cdots, m, \\ \bar{\tau}_{m+1} = \overset{*}{\tau}_{m+1} + \bar{\lambda}_1^{(h)} \overset{*}{\tau}_1 + \cdots + \bar{\lambda}_m^{(h)} \overset{*}{\tau}_m. \end{cases} \tag{11''}$$

Then $\overset{*}{\tau}_{m+1} = 0$ and this completes the proof.

II. Rationally independent values. The Cremona transformation $T^{(h)}$

3. We consider a zero-dimensional valuation B, *of rank 1*, of our field Σ. The residue field K' of B is an algebraic extension of the ground field K. To build up our Cremona transformations, we first consider the special case in which *the residue field* K' *coincides with the ground field.*

Let $x_1, \cdots, x_m$ be arbitrary elements of Σ, but such that $v(x_i) \geqq 0$. Since $\mathsf{K}' = \mathsf{K}$, there exist m uniquely determined constants $c_1, \cdots, c_m$ *in* K such that $v(x_i - c_i) > 0$. Let $v(x_i - c_i) = \tau_i$ and *let us assume that* $\tau_1, \cdots, \tau_m$ *are rationally independent.* We then associate with the Perron transformation (3) the following Cremona transformation $T^{(h)}$ in $\mathsf{K}(x_1, \cdots, x_m)$:

$$T^{(h)}\colon\ x_i - c_i = X_1^{A_i^{(h)}} X_2^{A_i^{(h+1)}} \cdots X_m^{A_i^{(h+m-1)}}, \quad i = 1, 2, \cdots, m. \tag{12}$$

$T^{(h)}$ is a Cremona transformation, since the determinant (4′) of the exponents $A_i^{(j)}$ is ± 1 (and is $+1$ if h is even). The x_i are polynomials in $X_1, X_2, \cdots, X_m$, since the $A_i^{(j)}$ are non-negative integers, by (4). Moreover, in view of (3) we have

$$v(X_i) = \tau_i^{(h)}. \tag{12'}$$

The fact that the ratios $A_i^{(h)}/A_1^{(h)}$ $(h = 1, 2, \cdots)$ are convergent fractions for τ_i/τ_1, will be used by us for the following purpose: we prove namely the following

Theorem 2. *Let*

$$\pi_1 = \prod_{i=1}^{m} (x_i - c_i)^{\mu_{i1}}, \cdots, \qquad \pi_N = \prod_{i=1}^{m} (x_i - c_i)^{\mu_{iN}},$$

be N power products of the differences $x_1 - c_1, \cdots, x_m - c_m$, and let

$$\pi_1 = \prod_{i=1}^{m} X_i^{\mathrm{M}_{i1}}, \cdots, \qquad \pi_N = \prod_{i=1}^{m} X_i^{\mathrm{M}_{iN}},$$

be the form which the elements $\pi_1, \cdots, \pi_N$ assume after the transformation $T^{(h)}$. If h is sufficiently high, then $\prod X_i^{\mathrm{M}_{i\alpha}}$ is a factor of $\prod X_i^{\mathrm{M}_{i\beta}}$ for all pairs of indices α and β such that $v(\pi_\alpha) < v(\pi_\beta)$.

Proof. We have

$$(13) \qquad \mathrm{M}_{ij} = \mu_{1j}A_1^{(h+i-1)} + \mu_{2j}A_2^{(h+i-1)} + \cdots + \mu_{mj}A_m^{(h+i-1)}.$$

If $v(\pi_\alpha) < v(\pi_\beta)$, then

$$(14) \qquad (\mu_{1\beta} - \mu_{1\alpha})\tau_1 + \cdots + (\mu_{m\beta} - \mu_{m\alpha})\tau_m > 0.$$

Since the ratios $A_i^{(h)}/A_1^{(h)}$ are convergent fractions for τ_i/τ_1, it is clear that if h is sufficiently high, say for $h \geqq h_0$, then also

$$(14') \qquad (\mu_{1\beta} - \mu_{1\alpha})A_1^{(h)} + \cdots + (\mu_{m\beta} - \mu_{m\alpha})A_m^{(h)} > 0,$$

for all pairs of indices $\alpha, \beta = 1, 2, \cdots, N$, such that (14) holds true. But (14′) implies $\mathrm{M}_{i\beta} > \mathrm{M}_{i\alpha}$, for $i = 1, 2, \cdots, m$, whenever $h \geqq h_0$, and this proves our theorem.

NOTE. Observe that for two *distinct* power products π_α and π_β it is not possible to have $v(\pi_\alpha) = v(\pi_\beta)$, since $\tau_1, \cdots, \tau_m$ are rationally independent.

4. We now consider the general case in which the residue field K' of our valuation B is an arbitrary algebraic extension of K, finite or infinite.[12] Let K^* be the least Galois extension of K which contains the field K'. For convenience we take K^* as our new ground field. By this ground field extension our original field Σ is extended to a field $\Sigma^* = \mathsf{K}^*\Sigma$, of degree of transcendency r over K^* (see [11], p. 189). It is well known that B admits at least one extension in Σ^* ([4], p. 185) and that K^* is the residue field of each extended valuation.[13] We

[12] The case of an infinite algebraic extension occurs already in the classical case, in view of our reduction of the dimensionality of a valuation through a transcendental extension of the ground field (see A III). The following example will illustrate. Let $\Sigma = \mathsf{K}(x, y, z)$, x, y and z—algebraically independent elements. Consider the one-dimensional valuation B obtained by equating z to the following formal power series in y:

$$z = x^{1/2}\cdot y + x^{1/4}y^2 + x^{1/8}y^3 + \cdots$$

Any element η in Σ becomes, by substitution, a formal power series in y, with coefficients which are algebraic functions of x, and by the value of η in B is meant the exponent of the leading term in the power series. Thus, for instance, $v(z) = 1$, $v(x) = 0$, etc. It is clear that the B-residue of z/y is $x^{1/2}$. We have $z^2/y^2 - x = \left(\frac{z}{y} - \sqrt{x}\right)\left(\frac{z}{y} + \sqrt{x}\right) = 2x^{1/2}x^{1/4}y + \cdots$, whence the B-residue of $\frac{z^2 - xy^2}{y^3}$ is $2x^{1/2}x^{1/4}$. Hence the B-residue of $\frac{z^2 - xy^2}{2zy^2}$ is $x^{1/4}$. Continuing in this fashion we can show that the residue field of B contains $x^{1/2}$, $x^{1/4}$, $x^{1/8}$, etc., i.e. the residue field is an infinite algebraic extension of the field $\mathsf{K}(x)$. But it is this last field which we would have to take as ground field if we wished to regard B as a zero-dimensional valuation.

[13] *If K' is an infinite extension of K, then B has necessarily infinitely many extension in Σ^*.* To see this, we first observe that a field of algebraic functions over a ground field K is by definition a finite extension of K. Hence the *relative algebraic closure* $\bar{\mathsf{K}}$ of K in Σ is also a finite extension of K. Therefore, if K' is an infinite extension of K, it is also an infinite extension of $\bar{\mathsf{K}}$. Therefore, for the proof of our assertion, we may assume that $\mathsf{K} = \bar{\mathsf{K}}$, or that K itself is maximally algebraic in Σ. In this case, the extended valuations of B form a *complete set of conjugate valuations* over K ([4], p. 185). In other words, if an

consider *one* of the extended valuations of B and we continue to denote it by B. However, when necessary we shall denote this extended valuation by B_1 in order to distinguish it from the other extended valuations $B_2, B_3, \cdots$.

Let $x_1, \cdots, x_m$ be elements of Σ contained in the valuation ring of B, and let the B-residue of x_i be α_i. We have then a point of coördinates $\alpha_1, \cdots, \alpha_m$ in the affine space of the x's (over the ground field K^*). Let $(\alpha_{1j}, \cdots, \alpha_{mj})$, $j = 1, 2, \cdots, g$, be the conjugate points over K. Here $(\alpha_{11}, \cdots, \alpha_{m1})$ is the original point $(\alpha_1, \cdots, \alpha_m)$.

Let $v(x_i - \alpha_i) = \tau_i$. *We make the following two assumptions*: (a) $\tau_1, \cdots, \tau_m$ are rationally independent; (b) the g conjugates $\alpha_{11} = \alpha_1, \alpha_{12}, \cdots, \alpha_{1g}$ are distinct.

We apply to $\tau_1, \cdots, \tau_m$ the elementary Perron transformation (1). Next, by using the interpolation formula of Lagrange, we construct a polynomial $\psi_i(x_1)$, $i > 1$, with coefficients in K, such that[14]

$$(15) \qquad \psi_i(x_1) - \alpha_{ij} \equiv 0(\text{mod } (x_1 - \alpha_{1j})^{a_i^{(0)}}), \qquad \begin{array}{l} j = 1, 2, \cdots, g; \\ i = 2, \cdots, m. \end{array}$$

Let $f(x_1) = \prod_{j=1}^{g} (x_1 - \alpha_{1j})$, whence $f(x)$ is in $\mathsf{K}[x_1]$ and is irreducible. We put

$$(16) \qquad \varphi_i(x_1) = \psi_i(x_1) + A_i(x_1)[f(x_1)]^{a_i^{(0)}},$$

where $A_i(x_1)$ is a polynomial in $\mathsf{K}[x_1]$ on which we will impose presently some conditions. However, whatever the polynomial $A_i(x_1)$, it is obvious that $\varphi_i(x_1)$ shares with $\psi_i(x_1)$ the property expressed by (15), i.e. we have

$$(17) \qquad \varphi_i(x_1) - \alpha_{ij} \equiv 0 \ (\text{mod } (x_1 - \alpha_{1j})^{a_i^{(0)}}), \qquad j = 1, 2, \cdots, g.$$

Let $f_j(x_1) = \dfrac{f(x_1)}{x_1 - \alpha_{1j}}$. By (15) we can write:

$$\frac{\psi_i(x_1) - \alpha_{ij}}{[f(x_1)]^{a_i^{(0)}}} = \frac{H_{ij}(x_1)}{[f_j(x_1)]^{a_i^{(0)}}},$$

where $H_{i1}, H_{i2}, \cdots, H_{ig}$ are polynomials in x_1 which are conjugate over K. Hence, by (16) we have

$$(17') \qquad \frac{\varphi_i(x_1) - \alpha_{ij}}{[f(x_1)]^{a_i^{(0)}}} = \frac{H_{ij}(x_1) + [f_j(x_1)]^{a_i^{(0)}} \cdot A_i(x_1)}{[f_j(x_1)]^{a_i^{(0)}}}.$$

Let us assign, for each $i \geqq 2$, an arbitrary set of g *distinct* conjugate elements

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element θ of K^* is a residue of an element of η in one extended valuation, each conjugate of θ over K is also a residue of η in some extended valuation. Suppose there were only a finite number, say g, of extended valuations. Then *every* element θ of K^* would have at most g conjugates over K, and this is impossible if K^* is an infinite extension of K.

[14] A polynomial $\psi_i(x_1)$ satisfying (15) exists and is uniquely determined if we impose the condition that it be of degree at most $a_i^{(0)}g - 1$.

$\alpha_{i-1,1}^{(1)}, \cdots, \alpha_{i-1,g}^{(1)}$ of the field $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$. We choose $A_i(x_1)$ (*in* $\mathsf{K}[x_1]$) in such a fashion as to satisfy the congruences

$$(18)\qquad [f_j(x_1)]^{a_i^{(0)}} A_i(x_1) \equiv -H_{ij}(x_1) - \alpha_{i-1,j}^{(1)} \cdot [f_j(x_1)]^{a_i^{(0)}} (\mathrm{mod}\ (x_1 - \alpha_{1j})^\rho), \qquad j = 1, 2, \cdots, g,$$

where ρ is a preassigned positive integer, which we can choose as high as we please. For $\rho = 1$, the existence of such a polynomial $A_i(x_1)$ follows directly from the interpolation formula of Lagrange and from the fact that $f_j(\alpha_{1j}) \neq 0$. For $\rho > 1$ the existence of $A_i(x_1)$ follows by an easy induction.[15]

5. Having set this condition on $A_i(x_1)$ and keeping in mind the definition (16) of the polynomial $\varphi_i(x_1)$, we now define our Cremona transformation $T^{(h)}$, *for* $h = 1$, as follows:

$$(19)\qquad T^{(1)}: \begin{cases} x_1 = x_m^{(1)}, \\ x_i = x_{i-1}^{(1)} \cdot [f(x_m^{(1)})]^{a_i^{(0)}} + \varphi_i(x_m^{(1)}), \qquad i = 2, \cdots, m. \end{cases}$$

From (17), (17′) and (18) we deduce that the equations of $T^{(1)}$ can also be written, for each $j = 1, 2, \cdots, g$, in the following form:

$$(19')\qquad T^{(1)}: \begin{cases} x_1 - \alpha_{1j} = x_m^{(1)} - \alpha_{mj}^{(1)}, \qquad \alpha_{1j} = \alpha_{mj}^{(1)} \\ x_i - \alpha_{ij} = (x_m^{(1)} - \alpha_{mj}^{(1)})^{a_i^{(0)}} \{[f_j(x_m^{(1)})]^{a_i^{(0)}} (x_{i-1}^{(1)} - \alpha_{i-1,j}^{(1)}) \\ \qquad\qquad + (x_m^{(1)} - \alpha_{mj}^{(1)})^\rho G_{ij}(x_m^{(1)})\}. \end{cases}$$

Here the G_{ij} are polynomials with coefficients in $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$. For optical reasons we re-write the equations (19′), dropping the index j referring to conjugate elements and denoting $f_j(x_m^{(1)})$ by $f^*(x_1)$:

$$(19'')\qquad \begin{aligned} x_1 - \alpha_1 &= x_m^{(1)} - \alpha_m^{(1)}, \qquad \alpha_1 = \alpha_m^{(1)}; \\ x_i - \alpha_i &= (x_m^{(1)} - \alpha_m^{(1)})^{a_i^{(0)}} \{[f^*(x_m^{(1)})]^{a_i^{(0)}} (x_{i-1}^{(1)} - \alpha_{i-1}^{(1)}) \\ &\qquad + (x_m^{(1)} - \alpha_m^{(1)})^\rho G_i(x_m^{(1)})\}, \quad i = 2, \cdots, m. \end{aligned}$$

We may, if we wish, regard the relations (19″) as referring to the index $j = 1$.

By (19), $x_1, \cdots, x_m$ are *polynomials* in $x_1^{(1)}, \cdots, x_m^{(1)}$, *with coefficients* in K. By (19″), the B-value of the expression in the curly brackets is equal to $\tau_i - a_i^{(0)}\tau_1$, i.e. to $\tau_{i-1}^{(1)}$. If we take ρ sufficiently large, so as to have $\rho\tau_1 > \tau_{i-1}^{(1)}$, then the value of the expression in the curly brackets will be the same as the value of the element $[f^*(x_m^{(1)})]^{a_i^{(0)}} \cdot (x_{i-1}^{(1)} - \alpha_{i-1}^{(1)})$. But $v(f^*(x_m^{(1)})) = v(f^*(x_1)) = 0$, since $f^*(x_1)$ is not divisible by $x_1 - \alpha_1$ (or, if we pay due respect to the index j: $f_j(x_1)$ is not divisible by $x_1 - \alpha_{1j}$). Hence, for such a large value of ρ we will have:

$$v(x_i^{(1)} - \alpha_i^{(1)}) = \tau_i^{(1)}, \qquad i = 1, 2, \cdots, m.$$

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[15] The existence of $A_i(x_1)$ also follows directly from a Lemma proved later on (B III 10, Lemma 1).

Thus the transformation $T^{(1)}$ has the desired effect, leading, as it does, to new elements $x_1^{(1)}, \cdots, x_m^{(1)}$, whose B-residues are $\alpha_1^{(1)}, \cdots, \alpha_m^{(1)}$ and such that the B-values of the differences $x_i^{(1)} - \alpha_i^{(1)}$ are the real numbers $\tau_i^{(1)}$ which are obtained at the first step of the algorithm of Perron. We emphasize that the conjugate elements $\alpha_{i1}^{(1)}, \alpha_{i2}^{(1)}, \cdots, \alpha_{ig}^{(1)}$ can be assigned, for each $i = 1, 2, \cdots, m$, arbitrarily; in particular they may be assumed to be distinct.

6. We now proceed to build up a Cremona transformation $T^{(h)}$ associated with the Perron transformation (3). Essentially $T^{(h)}$ will be obtained by a repeated application of the elementary transformation $T^{(1)}$ defined in the preceding section. We first give a full description of $T^{(h)}$ and we shall then prove its existence, by induction with respect to h.

First of all, $T^{(h)}$ will be of the form:

$$T^{(h)}: x_i = P_i(X_1, X_2, \cdots, X_m), \qquad i = 1, 2, \cdots, m, \tag{20}$$

where P_i is a polynomial with coefficients in K. The X_i belong to the valuation ring of B, and the B-residue of X_i *is an arbitrarily assigned element* γ_i of the field $\mathsf{K}(\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1g})$. Here, we recall, $\alpha_{11} = \alpha_1 = B$-residue of x_1 and $\alpha_{12}, \cdots, \alpha_{1g}$ are the conjugates of α_{11}. We also point out that by hypothesis these g conjugates are distinct, and consequently the field $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$ contains the B-residues of all the x_i, $i = 1, 2, \cdots, m$, and their conjugates over K. It is important to keep in mind the fact that the residues of X_i lie in the fixed field $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$, of relative degree g over K. That means that, as long as $\tau_1, \cdots, \tau_m$ are assumed to be rationally independent, the field generated by the coördinates of the centers of the extended valuations in the space of the X_i, and by their conjugates over K, remains fixed for any h. This fixed field $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$ we shall denote by R. If the center of B in the space of $x_1, \cdots, x_m$ splits into g conjugate points over K, also the center of B in the space of $X_1, \cdots, X_m$ will split into g and only g conjugate points over K. We have g = relative degree of R over K.

The above will be the first of a set of properties of $T^{(h)}$. We have then, as property (a), the following:

(a) $\gamma_i = B$-residue of X_i; γ_i—an arbitrary preassigned element of $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g})$. In particular, we may assume that γ_{i1} $(= \gamma_1)$ and its conjugates $\gamma_{i2}, \cdots, \gamma_{ig}$ over K are all distinct.

The other properties follow. For the sake of notations, we omit the index j relative to conjugate elements.[16]

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[16] In this connection one must bear in mind the following. The extended valuations of B in Σ^* form a complete set of conjugates under the Galois group of Σ^*/Σ ([4], p. 185). This Galois group coincides with the Galois group of K^*/K if and only if K is maximally algebraic in Σ ([11], p. 193). If then K is not maximally algebraic in Σ, there will certainly exist relative automorphisms of K^*/K which cannot be extended to relative automorphisms of Σ^*/Σ. In particular, it may well happen that a given relative automorphism π of R/K cannot be induced by a relative automorphism of Σ^*/Σ. (R may contain elements

(b) $v(X_i - \gamma_i) = \tau_i^{(h)}$.

(c) In the ring $\mathsf{R}[X_1, \cdots, X_m]$ the equations of $T^{(h)}$ can be put in the following form:

$$(20') \qquad T^{(h)}: x_i - \alpha_i = \Delta_1^{A_i^{(h)}} \Delta_2^{A_i^{(h+1)}} \cdots \Delta_m^{A_i^{(h+m-1)}} \cdot L_i(X_1, \cdots, X_m), \qquad i = 1, 2, \cdots, m.$$

Here $\Delta_1, \cdots, \Delta_m, L_i$ are polynomials with coefficients in R. The Δ_i are of the following form:

$$(21) \qquad \begin{aligned} \Delta_m &= X_m - \gamma_m, \\ \Delta_i &= A_i(X_{i+1}, \cdots, X_m)\cdot(X_i - \gamma_i) + (X_m - \gamma_m)^\rho H_i(X_{i+1}, \cdots, X_m), \\ & \qquad i = 1, 2, \cdots, m-1. \end{aligned}$$

Moreover,[17]

(d) $v(L_i) = v(A_i) = 0$ (whence $L_i(\gamma_1, \cdots, \gamma_m), A_i(\gamma_{i+1}, \cdots, \gamma_m) \neq 0$).

(e) $v(\Delta_i) = v(X_i - \gamma_i) = \tau_i^{(h)}$.

(f) ρ is an arbitrarily large integer.

7. Let us assume that a transformation $T^{(h)}$ with the above properties exists, for a given h. We prove that there also exists such a transformation $T^{(h+1)}$. We have, by (2),

$$\tau_1^{(h)} = \tau_m^{(h+1)}, \qquad \tau_i^{(h)} = \tau_{i-1}^{(h+1)} + a_i^{(h)} \tau_m^{(h+1)}, \qquad i = 2, \cdots, m.$$

With this elementary Perron transformation there is associated a Cremona transformation $\overline{T}^{(1)}$ defined in section 5:

$$(22) \qquad \overline{T}_1: \quad \begin{aligned} X_1 &= Y_m \\ X_i &= Y_{i-1}[f(Y_m)]^{a_i^{(h)}} + \varphi_i(Y_m), \end{aligned}$$

where $f(X_1) = \prod_{j=1}^{g} (X_1 - \gamma_{1j})$. The equations (22) are the same as (19), with a slight change of notations. By (19″), the equations of $\overline{T}_1$ assume in $\mathsf{R}[Y_1, \cdots, Y_m]$ the form:

$$(22') \qquad \begin{aligned} X_1 - \gamma_1 &= Y_m - \epsilon_m, \\ X_i - \gamma_i &= (Y_m - \epsilon_m)^{a_i^{(h)}} \{[f^*(Y_m)]^{a_i^{(h)}}\cdot(Y_{i-1} - \epsilon_{i-1}) + (Y_m - \epsilon_m)^\sigma G_i(Y_m)\}, \\ & \qquad i > 1, \end{aligned}$$

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which are not in K but are in Σ.) If this happens, and if, say, π carries the residues $\gamma_1, \cdots, \gamma_m$ ($= \gamma_{11}, \cdots, \gamma_{m1}$) into $\gamma_{12}, \cdots, \gamma_{m2}$, then $\gamma_{12}, \cdots, \gamma_{m2}$ *are not residues* of $X_1, \cdots, X_m$ for some extended valuation. To understand property (b) correctly it is therefore necessary to observe that it refers to our fixed extended valuation B_1 of B. If we apply the relative automorphisms of Σ^*/Σ, we have to write γ_{ij} instead of γ_i, *but j will not necessarily assume all the values* 1, 2, $\cdots$, g. At any rate, for all values which j actually assumes under these automorphisms, the symbol $v(X_i - \gamma_{ij})$ denotes the valuation-theoretic value of $X_i - \gamma_{ij}$ in a suitable extended valuation of B. Naturally, this value is always $\tau_i^{(h)}$, being independent of j.

[17] The remark concerning property (b) made in the preceding footnote also applies to properties (d) and (e).

where

(a′) ϵ_i = *B-residue of* Y_i ; $\epsilon_1, \cdots, \epsilon_{m-1}$, *arbitrary preassigned elements of* R, and $\epsilon_m = \gamma_1$. This corresponds to property (a).

Moreover,

$$f^*(X_1) = \frac{f(X_1)}{X_1 - \gamma_1}, \tag{23}$$

and σ is an arbitrarily large integer. We have also:

(b′) $v(Y_i - \epsilon_i) = \tau_i^{(h+1)}$.

We define: $T^{(h+1)} = T^{(h)} \cdot \bar{T}^{(1)}$. The transformation $T^{(h+1)}$, from $x_1, \cdots, x_m$ to $Y_1, \cdots, Y_m$, is such that the x_i are polynomials in $Y_1, \cdots, Y_m$ with coefficients in K, by (20) and (22). The conditions similar to (a) and (b) are satisfied, in view of (a') and (b').

If we substitute the expressions (22′) of the X_i into (21), we find that Δ_m is divisible by $(Y_m - \epsilon_m)^{a_m^{(h)}}$, and that also Δ_i, $1 < i < m$, acquires the factor $(Y_m - \epsilon_m)^{a_i^{(h)}}$, *provided* $\rho a_m^{(h)} - a_i^{(h)} > 0$. Now from (2′) it follows that if $h \geqq 1$, then $\tau_i^{(h)} < \tau_m^{(h)}$, whence $a_i^{(h)} \leqq a_m^{(h)}$, $i = 2, \cdots, m-1$. Hence by taking ρ sufficiently large, we can make $\rho a_m^{(h)} - a_i^{(h)}$ as large as we please. We therefore can put:

$$\Delta_i = (Y_m - \epsilon_m)^{a_i^{(h)}} \Delta'_{i-1}, \qquad i = 2, 3, \cdots, m, \tag{24}$$

and

$$\Delta_1 = (Y_m - \epsilon_m) \cdot M(Y_1, \cdots, Y_m). \tag{24′}$$

We find that $\Delta'_{i-1} - A_i(X_{i+1}, \cdots, X_m)[f^*(Y_m)]^{a_i^{(h)}}(Y_{i-1} - \epsilon_{i-1})$, $i = 2, \cdots, m-1$, is divisible by $(Y_m - \epsilon_m)^{\rho'}$, where $\rho' \geqq$ minimum $(\rho a_m^{(h)} - a_i^{(h)}, \sigma)$. Hence, by taking ρ and σ sufficiently large, we can make ρ' as large as we please. Let then

$$\begin{aligned} \Delta'_m &= Y_m - \epsilon_m \\ \Delta'_i &= A'_i(Y_{i+1}, \cdots, Y_m) \cdot (Y_i - \epsilon_i) + (Y_m - \epsilon_m)^{\rho'} H'_i(Y_{i+1}, \cdots, Y_m), \\ &\qquad\qquad i = 1, 2, \cdots, m-1, \end{aligned} \tag{25}$$

where

(f′) ρ' *is an arbitrarily large integer.* Here

$$A'_i(Y_{i+1}, \cdots, Y_m) = A_{i+1}(X_{i+2}, \cdots, X_m)[f^*(Y_m)]^{a_{i+1}^{(h)}} \qquad i = 1, 2, \cdots, m-2, \tag{26}$$

and, by (22′), for $i = m$,

$$A'_{m-1}(Y_m) = [f^*(Y_m)]^{a_m^{(h)}}. \tag{26′}$$

It is clear that $A_i^{(h+m)} = A_i^{(h)} + A_i^{(h+1)} a_2^{(h)} + \cdots + A_i^{(h+m-1)} a_m^{(h)}$. Hence substituting (24) and (24′) into (20′), and taking into account that by (25) $\Delta'_m = Y_m - \epsilon_m$, we get that the equations of $T^{(h+1)}$ in $\mathsf{R}[Y_1, \cdots, Y_m]$ are of the form:

$$x_i - \alpha_i = \Delta_1'^{A_i^{(h+1)}} \Delta_2'^{A_i^{(h+2)}} \cdots \Delta_m'^{A_i^{(h+m)}} \cdot L'_i(Y_1, \cdots, Y_m), \quad i = 1, 2, \cdots, m$$

where, by (24′),

$$(27) \qquad L_i'(Y_1, \cdots, Y_m) = L_i(X_1, \cdots, X_m) \cdot [M(Y_1, \cdots, Y_m)]^{A_i^{(h)}}.$$

These equations of $T^{(h+1)}$, together with the (25), give property (c).

From (24′) and from (21), for $i = 1$, we find that in $\mathsf{R}[Y_1, \cdots, Y_m]$ we have the following congruence:

$$M(Y_1, \cdots, Y_m) \equiv A_1(X_2, \cdots, X_m)(\text{mod } (Y_m - \epsilon_m)^{\rho a_m^{(h)}-1}).$$

Since $v(Y_m - \epsilon_m) = v(X_1 - \gamma_1) > 0$, and since $v(A_1) = 0$, it follows that $v(M) = 0$. Since $f^*(Y_m)$ is not divisible by $Y_m - \epsilon_m$ (by (23); note that $\gamma_{11} = \gamma_1 = \epsilon_m$), also $v(f^*(Y_m)) = 0$. Hence, in view of (26), (26′), (27) and of property (d), we conclude that $v(L_i') = v(A_i') = 0$, and this expresses property (d) for $T^{(h+1)}$. Finally, by (24), we have: $v(\Delta_i') = \tau_i^{(h+1)}$, for $i = 1, 2, \cdots, m-1$, and also $v(\Delta_m') = \tau_m^{(h+1)}$, since $\Delta_m' = Y_m - \epsilon_m = X_1 - \gamma_1$. Thus also property ($e$) holds true, and this completes the proof.

8. Theorem 2 admits an immediate generalization. If

$$\pi_1 = \prod_{i=1}^{m} (x_i - \alpha_i)^{\mu_{i1}}, \cdots, \qquad \pi_N = \prod_{i=1}^{m} (x_i - \alpha_i)^{\mu_{iN}},$$

are N distinct power products in the differences $x_i - \alpha_i$, after the transformation $T^{(h)}$ they become:

$$\pi_1 = G_1(X_1, \cdots, X_m) \prod_{i=1}^{m} \Delta_i^{\mathrm{M}_{i1}}, \cdots, \pi_N = G_N(X_1, \cdots, X_m) \prod_{i=1}^{m} \Delta_i^{\mathrm{M}_{iN}},$$

where the M_{ij} are given by the relations (13), and where $v(G_1) = \cdots = v(G_N) = 0$. Here $G_1, \cdots, G_N$ are polynomials with coefficients in the field R. *If h is sufficiently large, then we will have $\mathrm{M}_{i\beta} > \mathrm{M}_{i\alpha}$, $i = 1, 2, \cdots, m$, for all α and β such that $v(\pi_\beta) > v(\pi_\alpha)$.* The proof is the same as that of Theorem 2.

We make the following application of this result. Let $f(x_1, \cdots, x_m)$ be a polynomial with coefficients in K or in some algebraic extension field K_1 of K, for instance in K^*. Clearly, f is a linear combination of power products such as π_i, with coefficients in $(\mathsf{K}_1, \mathsf{R})$, say

$$f = c_1^* \pi_1 + \cdots + c_N^* \pi_N, \qquad c_i^* \neq 0.$$

If, for instance, π_1 is the power product of least value in B, then after the transformation $T^{(h)}$, if h is sufficiently large, f will acquire the factor $\Delta_1^{M_{11}} \cdots \Delta_m^{M_{m1}}$. If we divide through by this factor, what remains will be of the form $c_1^* G_1 +$ a multiple of the product $\Delta_1 \Delta_2 \cdots \Delta_m$. Since $v(G_1) = 0$ and $v(\Delta_1 \cdots \Delta_m) > 0$, we conclude with the following

THEOREM 3. *Given a polynomial $f(x_1, \cdots, x_m)$ with coefficients in K^*, if h is sufficiently high, the transform of f by $T^{(h)}$ will have the form*:

$$(28) \qquad f = \Delta_1^{M_1} \Delta_2^{M_2} \cdots \Delta_m^{M_m} F(X_1, \cdots, X_m),$$

where F is a polynomial and where $v(F) = 0$ (i.e. $F(\gamma_1, \cdots, \gamma_m) \neq 0$).

III. Rationally dependent values

9. Let $\tau_1, \cdots, \tau_m$ be rationally independent real numbers and let τ_{m+1} be rationally dependent on $\tau_1, \cdots, \tau_m$:

$$\lambda\tau_{m+1} = \lambda_1\tau_1 + \cdots + \lambda_m\tau_m, \qquad \lambda > 0, \qquad (\lambda, \lambda_1, \cdots, \lambda_m) = 1.$$

In that case we have Theorem 1 (12). *Let us assume that the residue field of a given zero-dimensional valuation B of Σ (of rank 1) coincides with the ground field K,* and let $x_1, \cdots, x_{m+1}$ be elements of Σ such that $v(x_i - c_i) = \tau_i$, where $c_i = B(x_1) \subset \mathsf{K}$. We associate with the transformation (6) of Theorem 1 the following Cremona transformation:

$$x_i - c_i = \prod_{j=1}^{m+1} X_j^{\beta_{ij}}, \qquad i = 1, 2, \cdots, m+1. \tag{29}$$

Then

$$\begin{aligned} v(X_i) &= \tau_i^* > 0, \qquad i = 1, 2, \cdots, m, \\ v(X_{m+1}) &= 0, \end{aligned} \tag{30}$$

where naturally $\tau_1^*, \cdots, \tau_m^*$ are rationally independent. The way in which we will apply the transformation (29) will be as follows. Let

$$\pi = \prod_{i=1}^{m+1} (x_i - c_i)^{\mu_i}, \qquad \pi' = \prod_{i=1}^{m+1} (x_i - c_i)^{\mu_i'},$$

be two power products of the differences $x_i - c_i$, where the μ_i and the μ_i' are non-negative integers, and let us assume that *π and π' have equal values in B*:

$$v(\pi) = v(\pi'). \tag{31}$$

After the transformation (29), the power products π and π' assume the form:

$$\pi = \prod_{i=1}^{m+1} X_i^{\mathrm{M}_i}, \qquad \pi' = \prod_{i=1}^{m+1} X_i^{\mathrm{M}_i'},$$

where

$$\begin{aligned} \mathrm{M}_i &= \mu_1\beta_{1i} + \cdots + \mu_{m+1}\beta_{m+1,i}, \\ \mathrm{M}_i' &= \mu_1'\beta_{1i} + \cdots + \mu_{m+1}'\beta_{m+1,i}. \end{aligned} \tag{32}$$

Since $v(\pi) = \mathrm{M}_1\tau_1^* + \cdots + \mathrm{M}_m\tau_m^*$ and $v(\pi') = \mathrm{M}_1'\tau_1^* + \cdots + \mathrm{M}_m'\tau_m^*$, and since $\tau_1^*, \cdots, \tau_m^*$ are rationally independent, it follows, by (31), that

$$\mathrm{M}_i = \mathrm{M}_i', \qquad i = 1, 2, \cdots, m. \tag{33}$$

In view of the relations (32) we have therefore the following equations:

$$\begin{aligned} \beta_{1i}(\mu_1' - \mu_1) + \cdots + \beta_{m+1,i}(\mu_{m+1}' - \mu_{m+1}) &= 0, \qquad i = 1, 2, \cdots, m \\ \beta_{1,m+1}(\mu_1' - \mu_1) + \cdots + \beta_{m+1,m+1}(\mu_{m+1}' - \mu_{m+1}) &= \mathrm{M}_{m+1}' - \mathrm{M}_{m+1}. \end{aligned} \tag{34}$$

The $(m + 1)$-rowed determinant $|\beta_{ij}|$ is ± 1. Hence, by (34), we have

$$\mathrm{M}'_{m+1} - \mathrm{M}_{m+1} = \frac{\mu'_{m+1} - \mu_{m+1}}{d}, \tag{35}$$

where d is, to within a $\pm$ sign, equal to the m-rowed determinant $|\beta_{ij}|$, $i, j = 1, 2, \cdots, m$.

10. In order to generalize the considerations of the preceding section to the general case of an arbitrary residue field, we first prove two lemmas.

Let R be a finite algebraic extension of the ground field K, of relative degree g. Let $P_j(\alpha_{1j}, \cdots, \alpha_{mj})$, $j = 1, 2, \cdots, g$, be g *distinct* conjugate points over K in the affine space $S_m(x_1, \cdots, x_m)(\alpha_{ij} \subset \mathsf{R})$. Let, moreover, $A_1(x_1, \cdots, x_m)$, $B_1(x_1, \cdots, x_m)$ be polynomials with coefficients in R, and let $A_j(x_1, \cdots, x_m)$, $B_j(x_1, \cdots, x_m)$ be the conjugate polynomials over K ($j = 1, 2, \cdots, g$).

Lemma 1. *If*

$$A_j(\alpha_{1j}, \cdots, \alpha_{mj}) \neq 0, \qquad j = 1, 2, \cdots, g, \tag{36}$$

then given any positive integer ρ_0, there exists a polynomial $H(x_1, \cdots, x_m)$ with coefficients in K such that $A_jH + B_j$, written as a polynomial in the differences $x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj}$, begins with terms of degree $\geqq \rho_0$.

Proof. Let $\mathfrak{p}_j$ denote the prime zero-dimensional ideal $(x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj})$ in the polynomial ring $\mathsf{R}[x_1, \cdots, x_m]$. Since the ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_g$ are distinct, any two of them are free from common divisors (*teilfremd*), and the same holds true for the powers $\mathfrak{p}_1^{\rho_0}, \cdots, \mathfrak{p}_g^{\rho_0}$. We therefore can find a polynomial ϕ_i in $\mathsf{R}[x_1, \cdots, x_m]$ such that

$$\phi_i \equiv 0(\mathfrak{p}_j^{\rho_0}), j \neq i; \qquad \phi_i \equiv 1(\mathfrak{p}_i^{\rho_0}), \tag{37}$$

and we may assume that $\phi_1, \cdots, \phi_g$ are conjugate polynomials over K.

Since, by hypothesis (36), $A_j \not\equiv 0(\mathfrak{p}_j)$, the ideal $(A_j, \mathfrak{p}_j^{\rho_0})$ is the unit ideal. Hence we can find a polynomial h_j in $\mathsf{R}[x_1, \cdots, x_m]$ such that $h_jA_j + B_j \equiv 0(\mathfrak{p}_j^{\rho_0})$, and again we may assume that $h_1, \cdots, h_g$ are conjugate polynomials over K. We put $H = h_1\phi_1 + \cdots + h_g\phi_g$. Then H is a polynomial with coefficients in K. Moreover, by (37), we have: $H \equiv h_j(\mathfrak{p}_j^{\rho_0})$, whence $HA_j + B_j \equiv h_jA_j + B_j \equiv 0(\mathfrak{p}_j^{\rho_0})$, q.e.d.

11. Let $\lambda_1, \cdots, \lambda_m$ be arbitrary non-negative integers.

Lemma 2. *Under the hypothesis that $\alpha_{11}, \cdots, \alpha_{1g}$ are distinct conjugates, given a set of conjugate elements $\gamma_1, \cdots, \gamma_g$ of R over K and given a positive integer ρ_0, there exists a polynomial $\varphi(x_1, \cdots, x_m)$ with coefficients in K and, for each $i = 1, 2, \cdots, m$, there exists a set of conjugate polynomials $\delta_{ij}(x_1, \cdots, x_i)$ ($j = 1, 2, \cdots, g$) with coefficients in R, such that:*

(38) $\varphi - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{mj}$;

$$\begin{aligned} (39) \qquad & \delta_{1j} = A_{1j}(x_1)\cdot(x_1 - \alpha_{1j})^{\lambda_1}, \\ & \delta_{ij} = A_{ij}(x_1, \cdots, x_i)\cdot(x_i - \alpha_{ij})^{\lambda_i} + B_{ij}(x_1, \cdots, x_i), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 2, \cdots, m; \end{aligned}$$

(40) *each* B_{ij} *begins with terms of degree* $\geqq \rho_0$ *in* $x_1 - \alpha_{1j}, \cdots, x_i - \alpha_{ij}$;

(40′) $A_{ij}(\alpha_{1j}, \cdots, \alpha_{ij}) \neq 0$;

(40″) $\delta_{ij}(\alpha_{1j'}, \cdots, \alpha_{ij'}) \neq 0$, if $j \neq j'$.

PROOF. For $m = 1$ the lemma follows directly from the interpolation formula of Lagrange. We assume the lemma to be true for a given m and for given $\lambda_1, \cdots, \lambda_m(\lambda_m \geqq 0)$, and we prove it for $m, \lambda_1, \cdots, \lambda_{m-1}, \lambda_m + 1$ (if $\lambda_m = 0$, we put $\delta_{mj} = 1$, and our induction is actually from $m - 1$ to m).

Let $d_i = \prod_{j=1}^{g} \delta_{ij}$ and $\delta^*_{ij} = d_i/\delta_{ij}$. Then d_i is a polynomial with coefficients in K. We put

$$\varphi' = \varphi + [cx_m + H(x_1, \cdots, x_{m-1})]\, d_1 d_2 \cdots d_m,$$

where c is some element of K and where H is a polynomial with coefficients in K, both to be determined. From (38) it follows that

$$(41) \qquad \varphi' - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{mj}\cdot\delta'_j,$$

where

$$(41') \qquad \delta'_j = 1 + [cx_m + H(x_1, \cdots, x_{m-1})]\delta^*_{1j} \cdots \delta^*_{mj}.$$

Let a_j and b_j be respectively the constant term and the coefficient of $x_m - \alpha_{mj}$ in the product $(\delta^*_{1j}\delta^*_{2j} \cdots \delta^*_{mj})$, when this polynomial is written as a polynomial in $x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj}$. In view of (40″), we have $a_j \neq 0$. We choose c in K so as to have $-b_j/a_j + ca_j \neq 0$. Having fixed the constant c, we find $H(x_1, \cdots, x_{m-1})$ in such a fashion that after the substitution $x_m = \alpha_{mj}$ the polynomial δ'_j, written as a polynomial in $x_1 - \alpha_{1j}, \cdots, x_{m-1} - \alpha_{m-1,j}$, begin with terms of degree $\geqq \rho_0$. This is possible, by Lemma 1, since $a_j \neq 0$. Then δ'_j has the following form:

$$\delta'_j = A_j(x_1, \cdots, x_m)\cdot(x_m - \alpha_{mj}) + B_j(x_1, \cdots, x_{m-1}),$$

where B_j begins with terms of degree $\geqq \rho_0$ in $x_1 - \alpha_{1j}, \cdots, x_{m-1} - \alpha_{m-1,j}$. Moreover, $A_j(\alpha_{1j}, \cdots \alpha_{mj}) \neq 0$, as a consequence of our choice of the constant c. It is clear that if we put $\delta'_{mj} = \delta_{mj}\delta'_j$, then δ'_{mj} will be of the same form as δ_{mj}, with $\lambda_m + 1$, instead of λ_m, and that the condition analagous to (40′) will be also satisfied. As to condition (40″), for δ'_{mj}, its validity follows from the definition (41′) of δ'_j : namely, we have obviously

$$(\delta'_j)_{x_i = \alpha_{ij'}} = 1, \qquad \text{if } j \neq j'.$$

Since, by (41), $\varphi' - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{m-1,j}\cdot\delta'_{mj}$, the proof is now complete.

$$\text{(35)} \qquad \mathrm{M}'_{m+1} - \mathrm{M}_{m+1} = \frac{\mu'_{m+1} - \mu_{m+1}}{d},$$

where d is, to within a $\pm$ sign, equal to the m-rowed determinant $|\beta_{ij}|$, $i, j = 1, 2, \cdots, m$.

10. In order to generalize the considerations of the preceding section to the general case of an arbitrary residue field, we first prove two lemmas.

Let R be a finite algebraic extension of the ground field K, of relative degree g. Let $P_j(\alpha_{1j}, \cdots, \alpha_{mj})$, $j = 1, 2, \cdots, g$, be g *distinct* conjugate points over K in the affine space $S_m(x_1, \cdots, x_m)(\alpha_{ij} \subset \mathsf{R})$. Let, moreover, $A_1(x_1, \cdots, x_m)$, $B_1(x_1, \cdots, x_m)$ be polynomials with coefficients in R, and let $A_j(x_1, \cdots, x_m)$, $B_j(x_1, \cdots, x_m)$ be the conjugate polynomials over K ($j = 1, 2, \cdots, g$).

LEMMA 1. *If*

$$\text{(36)} \qquad A_j(\alpha_{1j}, \cdots, \alpha_{mj}) \neq 0, \qquad j = 1, 2, \cdots, g,$$

then given any positive integer ρ_0, *there exists a polynomial* $H(x_1, \cdots, x_m)$ *with coefficients in* K *such that* $A_jH + B_j$, *written as a polynomial in the differences* $x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj}$, *begins with terms of degree* $\geqq \rho_0$.

PROOF. Let $\mathfrak{p}_j$ denote the prime zero-dimensional ideal $(x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj})$ in the polynomial ring $\mathsf{R}[x_1, \cdots, x_m]$. Since the ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_g$ are distinct, any two of them are free from common divisors (*teilfremd*), and the same holds true for the powers $\mathfrak{p}_1^{\rho_0}, \cdots, \mathfrak{p}_g^{\rho_0}$. We therefore can find a polynomial ϕ_i in $\mathsf{R}[x_1, \cdots, x_m]$ such that

$$\text{(37)} \qquad \phi_i \equiv 0(\mathfrak{p}_j^{\rho_0}), j \neq i; \qquad \phi_i \equiv 1(\mathfrak{p}_i^{\rho_0}),$$

and we may assume that $\phi_1, \cdots, \phi_g$ are conjugate polynomials over K.

Since, by hypothesis (36), $A_j \not\equiv 0(\mathfrak{p}_j)$, the ideal $(A_j, \mathfrak{p}_j^{\rho_0})$ is the unit ideal. Hence we can find a polynomial h_j in $\mathsf{R}[x_1, \cdots, x_m]$ such that $h_jA_j + B_j \equiv 0(\mathfrak{p}_j^{\rho_0})$, and again we may assume that $h_1, \cdots, h_g$ are conjugate polynomials over K. We put $H = h_1\phi_1 + \cdots + h_g\phi_g$. Then H is a polynomial with coefficients in K. Moreover, by (37), we have: $H \equiv h_j(\mathfrak{p}_j^{\rho_0})$, whence $HA_j + B_j \equiv h_jA_j + B_j \equiv 0(\mathfrak{p}_j^{\rho_0})$, q.e.d.

11. Let $\lambda_1, \cdots, \lambda_m$ be arbitrary non-negative integers.

LEMMA 2. *Under the hypothesis that* $\alpha_{11}, \cdots, \alpha_{1g}$ *are distinct conjugates, given a set of conjugate elements* $\gamma_1, \cdots, \gamma_g$ *of* R *over* K *and given a positive integer* ρ_0, *there exists a polynomial* $\varphi(x_1, \cdots, x_m)$ *with coefficients in* K *and, for each* $i = 1, 2, \cdots, m$, *there exists a set of conjugate polynomials* $\delta_{ij}(x_1, \cdots, x_i)$ ($j = 1, 2, \cdots, g$) *with coefficients in* R, *such that:*

$$\text{(38)} \qquad \varphi - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{mj};$$

(39) $$\delta_{1j} = A_{1j}(x_1)\cdot(x_1 - \alpha_{1j})^{\lambda_1},$$
$$\delta_{ij} = A_{ij}(x_1, \cdots, x_i)\cdot(x_i - \alpha_{ij})^{\lambda_i} + B_{ij}(x_1, \cdots, x_i),$$
$$i = 2, \cdots, m;$$

(40) *each* B_{ij} *begins with terms of degree* $\geqq \rho_0$ *in* $x_1 - \alpha_{1j}, \cdots, x_i - \alpha_{ij}$;
(40′) $A_{ij}(\alpha_{1j}, \cdots, \alpha_{ij}) \neq 0$;
(40″) $\delta_{ij}(\alpha_{1j'}, \cdots, \alpha_{ij'}) \neq 0$, if $j \neq j'$.

PROOF. For $m = 1$ the lemma follows directly from the interpolation formula of Lagrange. We assume the lemma to be true for a given m and for given $\lambda_1, \cdots, \lambda_m(\lambda_m \geqq 0)$, and we prove it for $m, \lambda_1, \cdots, \lambda_{m-1}, \lambda_m + 1$ (if $\lambda_m = 0$, we put $\delta_{mj} = 1$, and our induction is actually from $m - 1$ to m).

Let $d_i = \prod_{j=1}^{g} \delta_{ij}$ and $\delta^*_{ij} = d_i/\delta_{ij}$. Then d_i is a polynomial with coefficients in K. We put

$$\varphi' = \varphi + [cx_m + H(x_1, \cdots, x_{m-1})]\, d_1 d_2 \cdots d_m,$$

where c is some element of K and where H is a polynomial with coefficients in K, both to be determined. From (38) it follows that

(41) $$\varphi' - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{mj}\cdot\delta'_j,$$

where

(41′) $$\delta'_j = 1 + [cx_m + H(x_1, \cdots, x_{m-1})]\delta^*_{1j} \cdots \delta^*_{mj}.$$

Let a_j and b_j be respectively the constant term and the coefficient of $x_m - \alpha_{mj}$ in the product $(\delta^*_{1j}\, \delta^*_{2j} \cdots \delta^*_{mj})$, when this polynomial is written as a polynomial in $x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj}$. In view of (40″), we have $a_j \neq 0$. We choose c in K so as to have $-b_j/a_j + ca_j \neq 0$. Having fixed the constant c, we find $H(x_1, \cdots, x_{m-1})$ in such a fashion that after the substitution $x_m = \alpha_{mj}$ the polynomial δ'_j, written as a polynomial in $x_1 - \alpha_{1j}, \cdots, x_{m-1} - \alpha_{m-1,j}$, begin with terms of degree $\geqq \rho_0$. This is possible, by Lemma 1, since $a_j \neq 0$. Then δ'_j has the following form:

$$\delta'_j = A_j(x_1, \cdots, x_m)\cdot(x_m - \alpha_{mj}) + B_j(x_1, \cdots, x_{m-1}),$$

where B_j begins with terms of degree $\geqq \rho_0$ in $x_1 - \alpha_{1j}, \cdots, x_{m-1} - \alpha_{m-1,j}$. Moreover, $A_j(\alpha_{1j}, \cdots \alpha_{mj}) \neq 0$, as a consequence of our choice of the constant c. It is clear that if we put $\delta'_{mj} = \delta_{mj}\delta'_j$, then δ'_{mj} will be of the same form as δ_{mj}, with $\lambda_m + 1$, instead of λ_m, and that the condition analagous to (40′) will be also satisfied. As to condition (40″), for δ'_{mj}, its validity follows from the definition (41′) of δ'_j: namely, we have obviously

$$(\delta'_j)_{x_i = \alpha_{ij'}} = 1, \qquad \text{if } j \neq j'.$$

Since, by (41), $\varphi' - \gamma_j = \delta_{1j}\delta_{2j} \cdots \delta_{m-1,j}\cdot\delta'_{mj}$, the proof is now complete.

12. The significance of Theorem 3, as shown by (28), resides in the conclusion that the transform of a polynomial $f(x_1, \cdots, x_m)$, under $T^{(h)}$, if h is sufficiently high, differs from the transform of a power product of the differences $x_1 - \alpha_1, \cdots, x_m - \alpha_m$ (see (20′)) only by a *trivial factor*, i.e. by a factor whose value in B is zero. If, namely, $v(f) = \mu_1\tau_1 + \cdots + \mu_m\tau_m$, then the power product in question is $(x_1 - \alpha_1)^{\mu_1} \cdots (x_m - \alpha_m)^{\mu_m}$. In particular, if we have m arbitrary polynomials $\delta_1, \delta_2, \cdots, \delta_m$ in $\mathsf{K}^*[x_1, \cdots, x_m]$, such that $v(\delta_i) = \tau_i$, then, to within trivial factors, the δ_i will transform under $T^{(h)}$ like the differences $x_i - \alpha_i$, provided h is sufficiently high, i.e. we will have:

$$T^{(h)}\delta_i = \Delta_1^{A_i^{(h)}}\Delta_2^{A_i^{(h+1)}} \cdots \Delta_m^{A_i^{(h+m-1)}} \cdot L_i(X_1, \cdots, X_m), \tag{42}$$

where $v(L_i) = 0$. These relations are similar to (20′). More generally, a power product $\delta_1^{\mu_1} \cdots \delta_m^{\mu_m}$ will transform, to within trivial factors, as the product $(x_1 - \alpha_1)^{\mu_1} \cdots (x_m - \alpha_m)^{\mu_m}$; i.e., for h sufficiently high we will have:

$$T^{(h)}(\delta_1^{\mu_1} \cdots \delta_m^{\mu_m}) = \Delta_1^{\mathrm{M}_1} \cdots \Delta_m^{\mathrm{M}_m} G(X_1, \cdots, X_m), \tag{42′}$$

where $v(G) = 0$. While the real numbers $\tau_1, \cdots, \tau_m$ undergo the Perron transformation (3), it is clear that the exponents $\mu_1, \cdots, \mu_m$ undergo the *contragredient transformation* (13), i.e.:

$$\mathrm{M}_i = \mu_1 A_1^{(h+i-1)} + \cdots + \mu_m A_m^{(h+i-1)}), \qquad i = 1, 2, \cdots, m. \tag{42″}$$

Note that the polynomials $\Delta_1, \cdots, \Delta_m$ themselves form a particular set of polynomials such as $\delta_1, \cdots, \delta_m$, with respect, however, to the new variables $X_1, \cdots, X_m$. This is so because of property (e): $v(\Delta_i) = \tau_i^{(h)}$ (see B II 6).

More generally, suppose we consider a polynomial f which depends on $x_1, \cdots, x_m$ *and on other elements* $x_{m+1}, \cdots, x_n$ contained in the valuation ring $\mathfrak{B}$ of B. We write f as a polynomial in $x_1 - \alpha_1, \cdots, x_m - \alpha_m$, with coefficients which are polynomials in $x_{m+1}, \cdots, x_n$. Let $A(x_{m+1}, \cdots, x_n)(x_1 - \alpha_1)^{\mu_1} \cdots (x_m - \alpha_m)^{\mu_m}$ be the term for which $\mu_1\tau_1 + \cdots + \mu_m\tau_m$ is minimum. We shall say that *f is monovalent in* $x_1, \cdots, x_m$ if $v(A) = 0$. If f is monovalent, then $v(f) = \mu_1\tau_1 + \cdots + \mu_m\tau_m$, and it is obvious that under the transformation $T^{(h)}$, which affects only $x_1, \cdots, x_m$, f will transform like $(x_1 - \alpha_1)^{\mu_1} \cdots (x_m - \alpha_m)^{\mu_m}$, to within a trivial factor; provided h is sufficiently high. Similarly, if $\delta_1, \cdots, \delta_m$ are monovalent in $x_1, \cdots, x_m$ and if $v(\delta_i) = \tau_i$, then they also transform under $T^{(h)}$ according to (42); only the trivial factor L_i now depends on $X_1, \cdots, X_m$ *and* $x_{m+1}, \cdots, x_n$.

Remark. It is important to bear in mind that while the coefficients of the Cremona transformation $T^{(h)}$ are elements of K, the equations (20′) are the equations of $T^{(h)}$ written over the field R. Here R is the field generated by the B-residues of $x_1, \cdots, x_m$ and by their conjugates over K. Thus all the polynomials Δ_i, L_i in (20′) have their coefficients in R. As a consequence, if a given polynomial f in $x_1, \cdots, x_m$, and perhaps in other variables $x_{m+1}, \cdots, x_n$, has its coefficients in R, the coefficients of f, *after the transformation* $T^{(h)}$, *will remain in* R.

13. Let x_{m+1} be an element of Σ whose B-residue α_{m+1} is in the field R, i.e. in the field of the B-residues of $x_1, \cdots, x_m$ and of their conjugates. Let $v(x_{m+1} - \alpha_{m+1}) = \tau_{m+1} > 0$, and let us assume that τ_{m+1} is rationally dependent on $\tau_1, \cdots, \tau_m$:

$$\lambda\tau_{m+1} = \lambda_1\tau_1 + \cdots + \lambda_m\tau_m, \qquad \lambda > 0. \tag{43}$$

The transformation (6) of Theorem 1 consisted of two steps: the first step served to lower the value of λ [see (7), (10) and (7')], and at each stage of this step the transformations affected only rationally independent numbers. For $\lambda = 1$, we had the second step which consisted of a preliminary Perron transformation (which led to (11')) and of the transformation (11''). At this stage we shall be concerned with the first step. We wish to generalize in some way the considerations of the preceding section.

Let $\delta_1, \delta_2, \cdots, \delta_m$ be polynomials in $\mathsf{R}[x_1, \cdots, x_m]$ of the same form in $x_1, \cdots, x_m$ as the Δ_i's were in the X_i, i.e. (see (21)):

$$\begin{aligned} &\delta_m = x_m - \alpha_m, \\ (44) \qquad &\delta_i = a_i(x_{i+1}, \cdots, x_m)\cdot(x_i - \alpha_i) + (x_m - \alpha_m)^\sigma h_i(x_{i+1}, \cdots, x_m), \\ &\qquad\qquad i = 1, 2, \cdots, m-1, \end{aligned}$$

and with properties similar to (*d*), (*e*) and (*f*) (B II 6). Moreover, let

$$\delta_{m+1} = a_{m+1}(x_1, \cdots, x_m)\cdot(x_{m+1} - \alpha_{m+1}) + (x_m - \alpha_m)^\sigma h_{m+1}(x_1, \cdots, x_m) \tag{44'}$$

be another polynomial, of the indicated form, with coefficients in R. Such a set of $m + 1$ polynomials shall be called a *normal set* for $x_1, \cdots, x_m, x_{m+1}$, provided *$\sigma$ is sufficiently large*. How large must σ be will appear in the course of our considerations. *We also assume that, for each $i = 1, 2, \cdots, m$, the conjugates α_{i1} $(= \alpha_i)$, $\alpha_{i2}, \cdots, \alpha_{ig}$ are distinct* (compare with property (*a*) of B II 6).

We fix an integer ρ_0 such that

$$\rho_0\tau_m > \tau_i, \qquad i = 1, 2, \cdots, m-1, m+1. \tag{45}$$

Our *first* condition on σ will be:

$$\sigma \geqq \rho_0. \tag{45'}$$

We now pass by a Perron transformation from $\tau_1, \cdots, \tau_m$ to $\tau_1^{(h)}, \cdots, \tau_m^{(h)}$, getting (8) as the new relation of rational dependence between $\tau_1^{(h)}, \cdots, \tau_m^{(h)}$ and τ_{m+1}. We take h so high as to satisfy the following conditions:

(a) The coefficients $\lambda_1^{(h)}, \cdots, \lambda_m^{(h)}$ in (8) are all positive.

(b) $\lambda_m^{(h)} > \lambda$ (note that in view of (9), $\lim_{h\to+\infty} \lambda_i^{(h)} = +\infty$);

(c) $\lambda_m^{(h)} \not\equiv 0(\lambda)$. That it is possible to satisfy this condition can be seen as follows. The coefficients $\lambda_m^{(h)}, \lambda_m^{(h+1)}, \cdots, \lambda_m^{(h+m-1)}$, corresponding to m consecutive values of the upper index, are linear forms of $\lambda_1, \cdots, \lambda_m$, and the determinant of the coefficients of these forms is ± 1. Since $(\lambda_1, \cdots, \lambda_m, \lambda) = 1$,

at least one of the integers $\lambda_m^{(h)}, \lambda_m^{(h+1)}, \cdots, \lambda_m^{(h+m-1)}$ is not divisible by λ. If, say, $\lambda_m^{(h+i)} \not\equiv 0(\lambda)$, then we replace h by $h + i$.

(d) $\lambda\rho_0 A_m^{(h+m-1)} > \lambda_1 A_1^{(h+m-1)} + \cdots + \lambda_m A_m^{(h+m-1)}$, i.e. $\lambda\rho_0 A_m^{(h+m-1)} > \lambda_m^{(h)}$.

This inequality, for all high values of h, follows from the inequality (45): $\rho_0\tau_m > \tau_{m+1}$, and from (43).

(e) $\delta_1, \cdots, \delta_m$ *should transform under* $T^{(h)}$ *as* $x - \alpha_1, \cdots x_m - \alpha_m$ *respectively.* That this is the case, for sufficiently high values of h, follows from (45), for $i = 1, 2, \cdots, m - 1$, and from (45′). Namely, these inequalities tell us that $\delta_1, \cdots, \delta_m$ are monovalent in $x_1, \cdots, x_m$ and that $v(\delta_i) = \tau_i$. How high h must be in order to satisfy the condition (e) depends of course on the value of σ: the larger the value of σ is the sooner we will encounter satisfactory values of h. However, we wish to fix a lower bound for h which is *independent of* σ, since σ has not yet been fixed. For that it is sufficient to subject h to the following condition: h should be taken so high that $\delta_1, \cdots, \delta_m$ transform under $T^{(h)}$ as $x - \alpha_1, \cdots, x_m - \alpha_m$, *also if in* (44) *we replace* σ *by* ρ_0. This condition is stronger than condition (e), in view of (45′).

Now that h has been fixed we apply the transformation $T^{(h)}$ (x_{m+1} is so far not affected). *The integer* ρ *in* (21) *we take so high as to satisfy the condition*:

$$(45'') \qquad \rho\tau_1^{(h)} > \tau_i^{(h)}, \qquad i = 2, \cdots, m - 1.$$

The polynomials $\delta_1, \cdots, \delta_m$ are transformed according to (42). Instead of the set $\delta_1, \cdots, \delta_m, \delta_{m+1}$ we have now to deal with the polynomials $\Delta_1, \cdots, \Delta_m, \delta_{m+1}$. Concerning δ_{m+1} we only wish to observe that the factor $(x_m - \alpha_m)^\sigma$ in the second term of (44′) now takes the form:

$$(46) \qquad (x_m - \alpha_m)^\sigma = \Delta_1^{\sigma A_m^{(h)}} \Delta_2^{\sigma A_m^{(h+1)}} \cdots \Delta_m^{\sigma A_m^{(h+m-1)}} \cdot L,$$

where $\Delta_m = X_m - \gamma_m$ and where L is a trivial factor. Since, by (d) and (45′), $\sigma A_m^{(h+m-1)} > \dfrac{\lambda_m^{(h)}}{\lambda}$, it follows that if we put

$$(47) \qquad \lambda_m^{(h)} = g\lambda + \lambda', \qquad 0 < \lambda' < \lambda \qquad \text{(by (c))},$$

then

$$(48) \qquad (x_m - \alpha_m)^\sigma \equiv 0(\text{mod } (X_m - \gamma_m)^g).$$

We now operate on $\tau_m^{(h)}$ *and* τ_{m+1} *only.* We put, namely

$$(47') \qquad \tau_{m+1} = \bar{\tau}_{m+1} + g\tau_m^{(h)}, \qquad \bar{\tau}_{m+1} > 0, \text{ by (47)}.$$

For this elementary transformation involving *two* real numbers we have of course a Cremona transformation of the type $T^{(1)}$, from the variables X_m, x_{m+1} to the variables X_m, X_{m+1} (note that the conjugate residues $\gamma_{m1}, \gamma_{m2}, \cdots, \gamma_{m,g}$ of X_m are distinct, by property (a), B II 6). Written in the field R, the equation of the transformation $T^{(1)}$ is of the following form (compare with (19′)):

$$(49) \quad x_{m+1} - \alpha_{m+1} = (X_m - \gamma_m)^g\{C(X_m)\cdot(X_{m+1} - \gamma_{m+1}) + (X_m - \gamma_m)^{\bar{\rho}}\cdot D(X_m)\}.$$

Here we may assume that $\bar{\rho}$ is as high as we please and that the conjugate B-residues $\gamma_{m+1,1}\ (= \gamma_{m+1}), \gamma_{m+1,2}, \cdots, \gamma_{m+1,g}$ of X_{m+1} are distinct. These residues are in the field R.

By (48) and (49) δ_{m+1} acquires the factor $(X_m - \gamma_m)^g$. Let

(50) $$\delta_{m+1} = (X_m - \gamma_m)^g \Delta_{m+1} .$$

Then, in view of (46), (48), and (49),

(50′) $$\Delta_{m+1} \equiv A(X_1, \cdots, X_m)C(X_m)(X_{m+1} - \gamma_{m+1}) \bmod ((X_m - \gamma_m)^{\bar{\rho}}, \Delta_1^{\sigma A_m^{(h)}}),$$

and

(50″) $$v(AC) = 0.$$

We now have our new variables: $X_1, \cdots, X_m, X_{m+1}$, and our new set of polynomials: $\Delta_1, \cdots, \Delta_m, \Delta_{m+1}$ whose values are $\tau_1^{(h)}, \cdots, \tau_m^{(h)}$ and $\bar{\tau}_{m+1}$ respectively. The transformation (50) from δ_{m+1} to Δ_{m+1} is in an obvious relationship with the transformation (47′) from τ_{m+1} to $\bar{\tau}_{m+1}$.

We assume that $\bar{\rho}$ is so high as to satisfy the inequality:

(51) $$\bar{\rho}\tau_1^{(h)} > \bar{\tau}_{m+1} .$$

We now impose our second condition on the integer σ (in addition to (45′)):

(51′) $$\sigma A_m^{(h)} \tau_1^{(h)} > \bar{\tau}_{m+1} .$$

We know that $\tau_m^{(h)} > \tau_1^{(h)}$ (by (2)). Our final step will be an operation on $\tau_m^{(h)}$ and $\tau_1^{(h)}$; namely, we put:

$$\tau_m^{(h)} = \tau_1^{(h)} + \bar{\tau}_m .$$

At the same time we operate on X_1 and X_m by the corresponding Cremona transformation $T^{(1)}$, getting a new variable $\bar{X}_m$:

(52) $$X_m - \gamma_m = (X_1 - \gamma_1)\bar{\Delta}_m ,$$

where

(52′) $$\bar{\Delta}_m = \bar{C}(X_1)\cdot(\bar{X}_m - \bar{\gamma}_m) + (X_1 - \gamma_1)^{\rho'}\bar{D}(X_1), \qquad v(\bar{C}) = 0.$$

Here again ρ' can be as high as we please, and the conjugate B-residues $\bar{\gamma}_{m1}(= \bar{\gamma}_m), \bar{\gamma}_{m2}, \cdots, \bar{\gamma}_{mg}$ of $\bar{X}_m$ may be assumed to be distinct. They are elements of R, and so are the coefficients of $\bar{C}$ and $\bar{D}$.

From the expression (21) of Δ_1, we find:

53) $$\Delta_1 = (X_1 - \gamma_1)\cdot G(X_1, \cdots, X_{m-1}, \bar{X}_m), \qquad v(G) = 0.$$

We put

(53′) $$X_1 - \gamma_1 = \bar{\Delta}_1 .$$

Again from the expression (21) of the Δ_i, we deduce that each Δ_i, $i = 2, \cdots, m - 1$, *is monovalent in* X_i *and* X_1:

$$(54) \qquad \Delta_i \equiv \bar{A}_i(X_1, X_{i+1}, \cdots, X_{m-1}, \bar{X}_m)\cdot(X_i - \gamma_i) \bmod (X_1 - \gamma_1)^\rho,$$
$$v(\bar{A}_i) = 0, \qquad i = 2, \cdots, m-1.$$

Finally, as far as Δ_{m+1} is concerned, we deduce from (50′), (51), (51′), and (53) that Δ_{m+1} is monovalent in X_{m+1} and X_1. In conclusion, our new set of polynomials is the following: $\bar{\Delta}_1, \Delta_2, \cdots, \Delta_{m-1}, \Delta_{m+1}; \bar{\Delta}_m$. The polynomial Δ_1 differs from $\bar{\Delta}_1$ by a trivial factor $G(X_1, \cdots, X_{m-1}, \bar{X}_m)$ (see (53)), which, however, has this important property:

$$(55) \qquad G(\gamma_1, \cdots, \gamma_{m-1}, \bar{X}_m) = \text{const.}$$

This follows immediately from (53) and from the expression (21) of Δ_1. As a matter of fact, it is immediately seen that $G(\gamma_1, X_2, \cdots, X_{m-1}, \bar{X}_m)$ is a constant. As for $\bar{\Delta}_m$, we have, by (52): $\Delta_m = \bar{\Delta}_1\bar{\Delta}_m$, and this corresponds exactly to the relation $\tau_m^{(h)} = \tau_1^{(h)} + \bar{\tau}_m$. Finally, $\bar{\Delta}_1, \Delta_2, \cdots, \Delta_{m-1}, \Delta_{m+1}$ are monovalent in $X_1, X_2, \cdots, X_{m-1}, X_{m+1}$, and we have: $v(\bar{\Delta}_1) = \tau_1^{(h)}$, $v(\Delta_i) = \tau_i^{(h)}$ $i = 2, \cdots, m-1$; $v(\Delta_{m+1}) = \bar{\tau}_{m+1}$ and $v(\bar{\Delta}_m) = \bar{\tau}_m$. In the relation of rational dependence between $\tau_1^{(h)}, \cdots, \tau_{m-1}^{(h)}, \bar{\tau}_{m+1}$ and $\bar{\tau}_m$, the coefficient of $\bar{\tau}_m$ is λ', where $\lambda' < \lambda$ (see (47)).

14. Our result will become clearer if we change our notation. We denote $\tau_1^{(h)}$ by τ'_m; $\tau_i^{(h)}$ by τ'_i $(i = 2, \cdots, m-1)$; $\bar{\tau}_{m+1}$ by τ'_1 and $\bar{\tau}_m$ by τ'_{m+1}. Accordingly we denote $X_1, X_2, \cdots, X_{m-1}, \bar{X}_m, X_{m+1}$ by $x'_m, x'_2, \cdots, x'_{m-1}, x'_{m+1}, x'_1$ respectively and $\bar{\Delta}_1, \Delta_2, \cdots, \Delta_{m-1}, \bar{\Delta}_m, \Delta_{m+1}$ respectively by $\delta'_m, \delta'_2, \cdots, \delta'_{m-1}, \delta'_{m+1}, \delta'_1$. We denote the B-residue of x'_i by α'_i. We reassume our results by listing the following properties of the Cremona transformation from the x's to the x''s:

a). The variables $x_i (i = 1, 2, \cdots, m+1)$ are polynomials in $x'_1, \cdots, x'_{m+1}$ with coefficients in K. These polynomials reduce to constants, namely to $\alpha_1, \cdots, \alpha_{m+1}$, if $x'_1, \cdots, x'_m$ are replaced by their B-residues $\alpha'_1, \cdots, \alpha'_m$[18]

b). The conjugate B-residues $\alpha'_{i1}(= \alpha_i), \alpha'_{i2}, \cdots, \alpha_{ig}$ of x'_i are elements of R and may be assumed to be distinct.

c). If $\tau_i = \sum_{j=1}^{m+1} \beta_{ij}\tau'_j$, $i = 1, 2, \cdots, m+1$, are the equations of transformation of the values $\tau_1, \cdots, \tau_{m+1}$, then $\delta_i = G_i(x'_1, \cdots, x'_{m+1})\cdot\delta_1'^{\beta_{i1}} \cdots \delta_{m+1}'^{\beta_{i,m+1}}$.

d). $v(G_i) = 0$ and $G_i(\alpha'_1, \cdots, \alpha'_m, x'_{m+1})$ is a constant; the coefficients of G_i are in R.

e). The polynomials $\delta'_1, \cdots, \delta'_m$ are monovalent in $x'_1, \cdots, x'_m$. The polynomial δ'_{m+1} is of the form $C'(x'_m)\cdot(x'_{m+1} - \alpha'_{m+1}) + (x'_m - \alpha_m)^{\rho'} D'(x'_m)$, where $v(C') = 0$ and ρ' is as high as we please. The coefficients of $\delta'_1, \cdots, \delta'_{m+1}$ are in the field R.

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[18] In other words: all the points of the line $x'_1 = \alpha'_1, \cdots, x'_m = \alpha'_m$ in the space of $x'_1, \cdots, x'_m, x'_{m+1}$ correspond to the point $(\alpha_1, \cdots, \alpha_m, \alpha_{m+1})$ in the space of $x_1, \cdots, x_m, x_{m+1}$. Our assertion follows directly, by inspection, from the consecutive steps of the inverse Cremona transformation (from $X_1, \cdots, X_{m-1}, \bar{X}_m, X_{m+1}$ to $x_1, \cdots, x_{m+1}$).

f). $v(x_i' - \alpha_i') = v(\delta_i') = \tau_i'$.

g). We have the relations of rational dependence:
$\lambda\tau_{m+1} = \lambda_1\tau_1 + \cdots + \lambda_m\tau_m$ and $\lambda'\tau'_{m+1} = \lambda_1'\tau_1' + \cdots + \lambda_m'\tau_m'$, and in these relations λ' *is less than* λ.
We now operate on $x_1', \cdots, x_m'$ alone, by a transformation $T^{(h)}$. Under such a transformation, for h sufficiently high, the polynomials $\delta_1', \cdots, \delta_m'$ transform as $x_1' - \alpha_1', \cdots, x_m' - \alpha_m'$. It is clear that none of the above listed properties will be affected (in particular, the coefficient λ' in g) is not changed). The polynomials $\delta_1', \cdots, \delta_m'$ are replaced by new polynomials, which we shall continue to denote by $\delta_1', \cdots, \delta_m'$: they are of the same type in the new variables (which we shall continue to call $x_1', \cdots, x_m'$), as the polynomials $\Delta_1, \cdots, \Delta_m$, given in (21), are in $X_1, \cdots, X_m$. It is therefore clear that, provided we take the exponent ρ in (21) and the exponent ρ' in e) sufficiently high, the polynomials $\delta_1', \cdots, \delta_m', \delta_{m+1}'$ will form a *normal set.* Hence we may replace property e) by the following:

e'). *The polynomials* $\delta_1', \cdots, \delta_m', \delta_{m+1}'$ *form a normal set* (*with coefficients in* R).

Proceeding in this fashion we shall ultimately get a Cremona transformation which enjoys all the above properties and *such that in addition the coefficient* λ' *in* g) *is equal to* 1.

15. Suppose now that the coefficient λ was originally equal to 1:

$$\tau_{m+1} = \lambda_1\tau_1 + \cdots + \lambda_m\tau_m .$$

Let $\sigma_1, \cdots, \sigma_{m+1}$ be a normal set of polynomials in $\mathsf{R}[x_1, \cdots, x_m, x_{m+1}]$. We apply Lemma 2, where we put $\gamma_j = \alpha_{m+1,j}$. Let

$$(56)\qquad x'_{m+1} = \frac{x_{m+1} - \varphi}{d_1d_2 \cdots d_m} + H(x_1, \cdots, x_m), \qquad H \subset \mathsf{K}[x_1, \cdots, x_m],$$

where φ is the polynomial satisfying (38) and where $d_i = \prod_{j=1}^{g} \delta_{ij}$. In view of (38), we can rewrite (56) as follows:

$$(57)\qquad x'_{m+1} = \frac{x_{m+1} - \alpha_{m+1,j}}{d_1d_2 \cdots d_m} - \frac{1}{\delta_{1j}^* \cdots \delta_{mj}^*} + H,$$

where $\delta_{ij}^* = d_i/\delta_{ij}$.
In view of (40'') and of Lemma 1, we can find H in such a fashion that

$$(57')\qquad H\delta_{1j}^*, \cdots, \delta_{mj}^* - 1 = B_j(x_1, \cdots, x_m) = \varphi_{\sigma j}(x_1 - \alpha_{1j}, \cdots, x_m - \alpha_{mj}) + \text{terms of higher degree},$$

where $\varphi_{\sigma j}$ is a form of arbitrarily high degree σ.

Substituting into (57) we get

$$(57'')\qquad x_{m+1} - \alpha_{m+1,j} = \delta_{1j}\delta_{2j} \cdots \delta_{mj}(\delta_{1j}^* \cdots \delta_{mj}^* x'_{m+1} - B_j).$$

If ρ_0 in (40) is sufficiently high, then $v(\delta_{ij}) = \lambda_i \tau_i$, $i = 1, 2, \cdots, m$. Hence, by (57″),

$$v(\delta_{1j}^* \cdots \delta_{mj}^* x'_{m+1} - B_j) = 0,$$

and since $v(B_j) > 0$ (if $\sigma \geqq 1$), it follows that $v(x'_{m+1}) = 0$. By substitution of (57″), the polynomial σ_{m+1} will assume the following form (we denote δ_{i1} by δ_i)

$$\sigma_{m+1} = \delta_1 \delta_2 \cdots \delta_m \sigma'_{m+1} + (x_m - \alpha_m)^\sigma h(x_1, \cdots, x_m, x_{m+1}),$$

where

$$\sigma'_{m+1} = A(x_1, \cdots, x_m) x'_{m+1} + B(x_1, \cdots, x_m)$$

and $v(A) = 0$, $v(\sigma'_{m+1}) = 0$. In this form σ_{m+1} now appears as a monovalent polynomial in $x_1, \cdots, x_m$. Moreover, from (57″) it follows that if we replace $x_1, \cdots, x_m$ by their B-residues $\alpha_1, \cdots, \alpha_m$ (while letting x'_{m+1} be arbitrary), the expression of x_{m+1} as a polynomial in $x_1, \cdots, x_m, x_{m+1}$ reduces to a constant, namely to the B-residue α_{m+1} of x_{m+1}. If we now apply to $x_1, \cdots, x_m$ a Cremona transformation $T^{(h)}$, h sufficiently high, we obtain, in view of the results of the preceding section, the following

THEOREM 4. *Given a normal set of polynomials $\delta_1, \cdots, \delta_m, \delta_{m+1}$, there exists a unimodular integral non-negative transformation:* $\tau_i = \sum \beta_{ij} \tau_j^*$, *such that* $\tau_{m+1}^* = 0$; *and there exists a corresponding Cremona transformation*

$$x_i = P_i(X_1, \cdots, X_m, X_{m+1}), \qquad P_i \subset \mathsf{K}[X_1, \cdots, X_{m+1}], \tag{58}$$

where

$$P_i(\gamma_1, \cdots, \gamma_m, X_{m+1}) = \alpha_i, \quad \gamma_i = B \text{ residue of } X_i, \quad i = 1, 2, \cdots, m+1, \tag{58'}$$

such that

$$\delta_i = G_i(X_1, \cdots, X_m, X_{m+1}) \Delta_1^{\beta_{i1}} \cdots \Delta_m^{\beta_{im}} \Delta^{\beta_{i,m+1}}, \qquad i = 1, 2, \cdots, m+1,$$

and

(a) $\Delta_1, \cdots, \Delta_m$ *are of the form* (21);
(b) $v(X_i - \gamma_i) = v(\Delta_i) = \tau_i^*$, $i = 1, 2, \cdots, m$;
(c) $\Delta_{m+1} = A(X_1, \cdots, X_m) X_{m+1} + B(X_1, \cdots, X_m)$, $v(\Delta_{m+1}) = v(A) = 0$;
(d) $G_i(\gamma_1, \cdots, \gamma_m, X_{m+1}) = \text{const.}$
(e) *The residues γ_i of the elements $X_i (i = 1, 2, \cdots, m)$ and the coefficients of the Δ_i and of the $G_i (i = 1, 2, \cdots, m+1)$ are in the field* R.

It is important to point out that *the B-residue of X_{m+1} need not be in the field* R. Namely, from (57″) it follows that the B-residue of x'_{m+1} may very well be a proper algebraic quantity over R. Thus, of our various Cremona transformations, it is the last transformation, namely (56), that may lead for the first time to new variables whose B-residues generate a larger field than the one generated by the B-residues of the original variables $x_1, \cdots, x_{m+1}$.

From this theorem we can draw the same consequences as those given in B III 10. Namely, given two power products of $\delta_1, \cdots, \delta_{m+1}$:

$$\pi = \prod_{i=1}^{m+1} \delta_i^{\mu_i}, \qquad \pi' = \prod_{i=1}^{m+1} \delta_i^{\mu_i'},$$

and assuming that $v(\pi) = v(\pi')$, then after the transformation (58) has been performed, π and π' assume the form:

$$\pi = G(X_1, \cdots, X_m, X_{m+1}) \prod_{i=1}^{m+1} \Delta_i^{\mathrm{M}_i},$$

$$\pi' = G'(X_1, \cdots, X_m, X_{m+1}) \prod_{i=1}^{m+1} \Delta_i^{\mathrm{M}_i'},$$

where $G(\gamma_1, \cdots, \gamma_m, X_{m+1})$ and $G'(\gamma_1, \cdots, \gamma_m, X_{m+1})$ are *constants* different from zero, and where the exponents M_i, M_i' satisfy the relations (33) and (35).

C. Uniformization of Valuations

I. Zero-dimensional valuations of rank 1. A lemma

1. The proof of the main theorem is based on a lemma concerning rank 1 zero-dimensional valuations in a field $\mathsf{K}(x_1, \cdots, x_r, x_{r+1})$, where $x_1, \cdots, x_{r+1}$ are *algebraically independent* elements. Let B be such a valuation, with K' as residue field. Let $\alpha_1, \cdots, \alpha_{r+1}$ be the B-residues of $x_1, \cdots, x_{r+1}$ respectively (we assume that $v(x_i) \geqq 0$), and let R be the field generated by these residues and by their conjugates over K. Finally, let K^* be the least Galois extension of K containing K'.

Lemma. *Given a polynomial $f(x_1, \cdots, x_{r+1})$, with coefficients in R, there exists a Cremona transformation*:

$$x_i = P_i(X_1, \cdots, X_{r+1}), \qquad i = 1, 2, \cdots, r+1,$$

where the P_i are polynomials with coefficients in K, such that the expression of f in terms of the new variables X_i is of the form:

$$f = \Delta_1^{\mu_1} \cdots \Delta_m^{\mu_m} F(X_1, \cdots, X_{r+1}), \tag{59}$$

where:

(a) $v(X_i) \geqq 0, i = 1, 2, \cdots, r+1$;

(b) $v(\Delta_i) = v(X_i - \gamma_i)$, *where γ_i is the B-residue of X_i* ;

(c) $v(\Delta_1), \cdots, v(\Delta_m)$ *are rationally independent numbers*;

(d) $\Delta_1, \cdots, \Delta_m$ *are polynomials in $X_1, \cdots, X_m$, of the form* (21); *the coefficients of the Δ_i and of F are in the field generated by $\gamma_1, \cdots, \gamma_{r+1}$ and by their conjugates over* K. *If $\mathsf{K}' = \mathsf{K}$, then $\Delta_i = X_i$.*

(e) $v(F) = 0$.

It should be understood that the values $v(\Delta_i)$, $v(F)$ are taken relatively to some extended valuation of B in $\mathsf{K}^*\Sigma$.

Main theorem and Lemma will be proved simultaneously, *under the hypothesis that the lemma is true in the case of r independent variables.*

In the case $r = 1$, the lemma is trivial. In fact, let $\alpha_1 = B(x_1)$. If $v(f(x_1)) = 0$, there is nothing to prove. If $v(f(x_1)) > 0$, then $f(x_1)$ is divisible by $x_1 - \alpha_1$. If $f(x_1)$ is exactly divisible by $(x_1 - \alpha_1)^{\mu_1}$, then the factorization $f(x_1) = (x_1 - \alpha_1)^{\mu_1} F(x_1)$ yields (59).

II. Special case: residue field = ground field

2. We shall first treat the case in which the residue field of B coincides with the ground field K. Let $f(x_1, x_2, \cdots, x_{r+1})$ be the given polynomial with coefficients in K. In the case of the main theorem, f is irreducible and $f = 0$ is the defining equation of our hypersurface. In the case of the lemma, $x_1, \cdots, x_{r+1}$ are algebraically independent elements. Without loss of generality we may assume that $v(x_i) > 0$, $i = 1, 2, \cdots, r + 1$, i.e. that the point $(0, \cdots, 0)$ is the center of the valuation. We write f as a polynomial in x_{r+1}:

$$f = a_0(x_1, \cdots, x_r) + a_1(x_1, \cdots, x_r)x_{r+1} + \cdots + a_\nu(x_1, \cdots, x_r)x_{r+1}^\nu .$$

In the case of the theorem we assume that $f(0, \cdots, 0, x_{r+1}) \neq 0$, i.e. that our hypersurface does not contain the line $x_1 = \cdots = x_r = 0$. In the case of the lemma, if $h(x_1, \cdots, x_r)$ is the h.c.d. of $a_0, \cdots, a_\nu$, and if we put $f/h = \varphi$, we assume that $\varphi(0, \cdots, 0, x_{r+1}) \neq 0$. These assumptions can always be satisfied if we perform a preliminary linear homogeneous transformation on $x_1, \cdots, x_{r+1}$.

Whether we are dealing with the theorem or with the lemma, it is true in both cases that $x_1, \cdots, x_r$ are algebraically independent elements. Hence, by our induction, we can apply the lemma to the product $a_0a_1 \cdots a_\nu$ and to the valuation induced by B in the field $\mathsf{K}(x_1, \cdots, x_r)$. There exists then a Cremona transformation: $x_i = P_i(X_1, \cdots, X_r)$, $i = 1, 2, \cdots, r$, such that each polynomial a_i assumes the following form:

$$a_i(x_1, \cdots, x_r) = X_1^{\mu_{1i}} \cdots X_m^{\mu_{mi}} b_i(X_1, \cdots, X_r),$$

where $\mu_{1i}, \cdots, \mu_{mi}$ are non-negative integers and $v(b_i) = 0$. Moreover, $v(X_i) \geqq 0$, $i = 1, 2, \cdots, r$, and $v(X_1), \cdots, v(X_m)$ are rationally independent numbers. We assume that this Cremona transformation has already been performed, and we identify the X's with the x's. For convenience, we denote $X_{m+1}, \cdots, X_r$ and X_{r+1} by $x_{m+2}, \cdots, x_{r+1}$ and x_{m+1} respectively. Then f has the following form:

$$f = b_0x_1^{\mu_{10}} \cdots x_m^{\mu_{m0}} + b_1x_1^{\mu_{11}} \cdots x_m^{\mu_{m1}}x_{m+1} + \cdots + b_\nu x_1^{\mu_{1\nu}} \cdots x_m^{\mu_{m\nu}}x_{m+1}^\nu , \tag{60}$$

where $b_0, b_1, \cdots, b_\nu$ are polynomials in $x_1, \cdots, x_m, x_{m+2}, \cdots, x_{r+1}$, and $v(b_i) = 0$.

Let $v(x) = \tau_i$, $i = 1, 2, \cdots, m + 1$. We have two possible cases: (1) $\tau_1, \cdots, \tau_{m+1}$ are rationally independent; (2) they are rationally dependent.

In the first case we are dealing necessarily with the lemma, since in the case of

the theorem the defining equation $f = 0$ implies, in view of (60), that τ_{m+1} is rationally dependent on $\tau_1, \cdots, \tau_m$ (if $f = 0$, at least two of the $\nu + 1$ terms of f must have the same value in B). In this case the lemma follows directly from Theorem 2 (B II 3). Namely, if we replace in Theorem 2 the integer m by $m + 1$ and if we identify the power product π_i with $x_1^{\mu_{1i}} \cdots x_m^{\mu_{mi}} x_{m+1}^m$, $i = 0, 1, 2, \cdots, \nu$, then under our Cremona transformation $T^{(h)}$, applied to $x_1, \cdots, x_{m+1}$, h sufficiently high, f will assume the form:

$$f = X_1^{M_{1\alpha}} \cdots X_{m+1}^{M_{m+1,\alpha}}[b_\alpha + X_1 X_2 \cdots X_{m+1} H(X_1, \cdots, X_{r+1})],$$

$$X_i = x_i, \; i = m + 2, \cdots, r + 1,$$

where we assume that π_α is the term of lowest value in B. Since $v(b_\alpha) = 0$ and $v(X_i) > 0$, our lemma is proved.

3. We now consider the case in which $\tau_1, \cdots, \tau_{m+1}$ are rationally dependent. In the case of the theorem we have assumed that $f(0, \cdots, 0\ x_{r+1}) \neq 0$ and also that $f(0, \cdots, 0, 0) = 0$, since, by assumption, $v(x_i) > 0$, $i = 1, 2, \cdots, r + 1$. In the case of the lemma we have assumed that $\varphi(0, \cdots, 0, x_{r+1}) \neq 0$, where $f = h(x_1, \cdots, x_r) \cdot \varphi(x_1, \cdots, x_{r+1})$, and where h is the h.c.d. of $a_0, \cdots, a_\nu$. Let s be the multiplicity of the root $x_{r+1} = 0$ for the polynomial $f(0, \cdots, 0, x_{r+1})$, in the case of the theorem; for the polynomial $\varphi(0, \cdots, 0, x_{r+1})$, in the case of the lemma. Then $s \geqq 1$, in the first case; $s \geqq 0$ in the second case. In the case of the theorem, if $s = 1$, then the center $(0, \cdots, 0)$ of B is already a simple point of our hypersurface, and there is nothing to prove. In the case of the lemma, if $s = 0$, then $v(\varphi) = 0$ and, by our induction, the proof of the lemma is completed by applying the lemma to the polynomial h *of the r variables* $x_1, \cdots, x_r$. We proceed to show that it is always possible to achieve a reduction of the multiplicity s, by Cremona transformations of the desired type, as long as $s > 1$ or $s > 0$, according as we are dealing with the theorem or with the lemma.

4. To the rationally dependent numbers $\tau_1, \cdots, \tau_m, \tau_{m+1}$ we apply the transformation (6) of Theorem 1, and we operate on $x_1, \cdots, x_m, x_{m+1}$ by the corresponding Cremona transformation (29), where we now put $c_i = 0$ (B III 9). Let

$$\pi_i = b_i x_1^{\mu_{1i}} \cdots x_m^{\mu_{mi}} x_{m+1}^i, \qquad i = 0, 1, \cdots, \nu,$$

and let $\pi_\alpha, \pi_\beta \cdots, \pi_\delta$, $\alpha < \beta < \cdots < \delta$ be the terms of *lowest value*. Then

$$\pi_i = b_i X_1^{\mathbf{M}_{1i}} \cdots X_m^{\mathbf{M}_{mi}} X_{m+1}^{\mathbf{M}_{m+1,i}},$$

where the $\mathbf{M}_{ji}$ are given by relations similar to (32), in which, however, $\mu_{m+1,i}$ should be replaced by i. In view of (33), the polynomial f assumes the form:

$$\begin{aligned} f = X_1^{\mathbf{M}_1} \cdots X_m^{\mathbf{M}_m}[B_\alpha X_{m+1}^{\mathbf{M}_{m+1,\alpha}} + B_\beta X_{m+1}^{\mathbf{M}_{m+1,\beta}} + \cdots + B_\delta X_{m+1}^{\mathbf{M}_{m+1,\delta}}] \\ + \sum{}' B_i X_1^{\mathbf{M}_{1i}} \cdots X_m^{\mathbf{M}_{mi}} X_{m+1}^{\mathbf{M}_{m+1,i}}, \end{aligned} \tag{61}$$

where $\mathrm{M}_i = \mathrm{M}_{i\alpha} = \mathrm{M}_{i\beta} = \cdots = \mathrm{M}_{i\delta}$ $(i = 1, 2, \cdots, m)$. The summation symbol $\sum'$ is extended to all terms π_i other than the minimum value terms. Moreover, $B_i(X_1, \cdots, X_{m+1}, x_{m+2}, \cdots, x_{r+1}) = b_i(x_1, \cdots, x_{r+1})$. We know that $v(X_{m+1}) = 0$. Hence, by (29), it follows that, for a given $i = 1, 2, \cdots, m + 1$, the exponents $\beta_{i1}, \cdots, \beta_{im}$ cannot be all zero. Hence

$$B_i(0, \cdots, 0, X_{m+1}, 0, \cdots, 0) = b_i(0, \cdots, 0). \tag{61'}$$

By (35) we have

$$\begin{aligned} \mathrm{M}_{m+1,\beta} - \mathrm{M}_{m+1,\alpha} &= \frac{\beta - \alpha}{d}, \\ &\cdots\cdots\cdots\cdots \\ \mathrm{M}_{m+1,\delta} - \mathrm{M}_{m+1,\alpha} &= \frac{\delta - \alpha}{d}. \end{aligned} \tag{62}$$

According as $d > 0$ or $d < 0$, the power $X_{m+1}^{\mathrm{M}_{m+1,\alpha}}$, or the power $X_{m+1}^{\mathrm{M}_{m+1,\delta}}$ will factor out from all the terms in the square brackets in (61), i.e. we shall have either

$$\begin{aligned} &\pi_\alpha + \pi_\beta + \cdots + \pi_\delta \\ &\quad = X_1^{\mathrm{M}_1} \cdots X_m^{\mathrm{M}_m} X_{m+1}^{\mathrm{M}_{m+1,\alpha}} [B_\alpha + B_\beta X_{m+1}^{(\beta-\alpha)/d} + \cdots + B_\delta X_{m+1}^{(\delta-\alpha)/d}], \end{aligned}$$

or

$$\begin{aligned} &\pi_\alpha + \pi_\beta + \cdots + \pi_\delta \\ &\quad = X_1^{\mathrm{M}_1} \cdots X_m^{\mathrm{M}_m} X_{m+1}^{\mathrm{M}_{m+1,\delta}} [B_\alpha X_{m+1}^{(\alpha-\delta)/d} + B_\beta X_{m+1}^{(\beta-\delta)/d} + \cdots + B_\delta], \end{aligned}$$

according as $d > 0$ or $d < 0$.

Since $\pi_\alpha, \pi_\beta, \cdots, \pi_\delta$ were the lowest value terms, we have $M_1\tau_1^* + \cdots + M_m\tau_m^* < M_{1i}\tau_1^* + \cdots + M_{mi}\tau_m^*$, for all i such that π_i is not a lowest value term. We now operate on $X_1, \cdots, X_m$ by a Cremona transformation $T^{(h)}$, h sufficiently high, and we apply Theorem 2. Let $y_1, \cdots, y_m$ be the new variables (in Theorem 2 the new variables were denoted by $X_1, X_2, \cdots$). For convenience we denote by $y_{m+1}, y_{m+2}, \cdots, y_{r+1}$ the variables $X_{m+1}, x_{m+2}, \cdots, x_{r+1}$. Then the polynomial f assumes the following form:

$$f = y_1^{\Lambda_1} \cdots y_m^{\Lambda_m} F(y_1, \cdots, y_{r+1}), \tag{63}$$

where

$$F = y_{m+1}^{\Lambda_{m+1}} \psi(y_1, \cdots, y_{r+1}) + y_1 y_2 \cdots y_m G(y_1, \cdots, y_{r+1}), \ G, \text{ a polynomial.} \tag{63'}$$

Here

$$\psi = A_\alpha + A_\beta y_{m+1}^{(\beta-\alpha)/d} + \cdots + A_\delta y_{m+1}^{(\delta-\alpha)/d} \text{ and } \Lambda_{m+1} = \mathrm{M}_{m+1,\alpha} \text{ if } d > 0, \tag{64}$$

and

$$\psi = A_\alpha y_{m+1}^{(\alpha-\delta)/d} + A_\beta y_{m+1}^{(\beta-\delta)/d} + \cdots + A_\delta \text{ and } \Lambda_{m+1} = \mathrm{M}_{m+1,\delta} \text{ if } d < 0. \tag{64'}$$

Moreover

$$A_i(y_1, \cdots, y_{r+1}) = B_i(X_1, \cdots, X_m, X_{m+1}, x_{m+2}, \cdots, x_{r+1}),$$

whence, by (61′),

$$A_i(0, \cdots, 0, y_{m+1}, 0, \cdots, 0) = b_i(0, \cdots, 0) \neq 0. \tag{65}$$

In the case of the theorem, the defining equation of the transformed hypersurface is either $F = 0$ or is $F_1 = 0$, where F_1 is a factor of F.

Let c be the B-residue of y_{m+1}. Then $c \neq 0$, since $v(X_{m+1}) = v(y_{m+1}) = 0$. By (65) it follows that ψ does not vanish identically if we put $y_i = 0, i \neq m + 1$. Hence, by (63′), the same holds true for the polynomial F. Let $y_{m+1} = c$ be a root of multiplicity $s'(s' \geqq 0)$ for the polynomial $F(0, \cdots, 0, y_{m+1}, 0, \cdots, 0)$. From (64), (64′) and (65) it follows that $s' \leqq (\delta - \alpha)/|d|$. Let us compare $(\delta - \alpha)/|d|$ with the original multiplicity s. We have $\pi_s = a_s(x_1, \cdots, x_m, x_{m+2}, \cdots, x_{r+1}) \cdot x_{m+1}^s$. In the case of the theorem, we must have $a_s(0, \cdots, 0) \neq 0$, since $x_{m+1} = 0$ is an s-fold root of $f(0, \cdots, 0, x_{m+1}, 0, \cdots, 0)$. In the case of the lemma, $f = h\varphi$ and $a_s = h\bar{a}_s$, where h is the h.c.d. of $a_0, a_1, \cdots, a_\nu$ and again we must have $\bar{a}_s(0, \cdots, 0) \neq 0$. In either case we conclude that $v(\pi_i) > v(\pi_s)$ for all $i > s$, since $v(\pi_i) \geqq v(x_{m+1}^i)$ and $v(\pi_s) = v(x_{m+1}^s)$. Since $\pi_\alpha, \pi_\beta, \cdots, \pi_\delta$ are the lowest value terms, it follows that

$$0 \leqq \alpha < \delta \leqq s. \tag{66}$$

Hence

$$s' \leqq \frac{|\delta - \alpha|}{|d|} \leqq s. \tag{66′}$$

Let us suppose that there is no reduction in the value of s, i.e. that $s' = s$. (Note that y_{m+1} plays now the role of x_{m+1}, or, in our old notations, of x_{r+1}). This implies in the first place $(\delta - \alpha)/|d| = s$, by (66′), whence, by (66), $\alpha = 0$ and $\delta = s$, $|d| = 1$. In the second place, we must have that $\psi(0, \cdots, 0, y_{m+1}, 0, \cdots, 0)$, to within a constant factor, is equal to $(y_{m+1} - c)^s$. This implies that $\alpha = 0, \beta = 1, \cdots, \delta = s$, i.e. originally there must have been in the polynomial f terms π_i of degree i in x_{m+1}, for all $i = 0, 1, \cdots, s$, and these terms were the lowest value terms. In particular, we have the terms $\pi_{s-1} = a_{s-1}x_{m+1}^{s-1}$ and $\pi_s = a_s x_{m+1}^s$, and since $v(\pi_{s-1}) = v(\pi_s)$, it follows that $v(a_{s-1}) = v(a_s) + v(x_{m+1})$. In the case of the theorem we have $a_s(0, \cdots, 0) \neq 0$, whence $v(x_{m+1}) = v(a_{s-1})$. Similarly, in the case of the lemma, we have $v(\bar{a}_{s-1}) = v(\bar{a}_s) + v(x_{m+1})$ and $v(\bar{a}_s) = 0$, whence $v(x_{m+1}) = v(\bar{a}_{s-1})$. In either case we conclude that *if the multiplicity s is not reduced by our Cremona transformation (from the x's to the y's), then the value of x_{m+1} must be equal to the value of a polynomial in the remaining variables $x_1, \cdots, x_m, x_{m+2}, \cdots, x_{r+1}$.*

5. For convenience we go back to our original notations in which x_{r+1} was the variable which latter was denoted by x_{m+1}. We assume that our Cremona

transformation failed to achieve a reduction of the multiplicity s, and that consequently

$$v(x_{r+1}) = v(H), \qquad H \subset K[x_1, \cdots, x_r].$$

Since $v\left(\frac{x_{r+1}}{H}\right) = 0$, there exists a constant $c \neq 0$ in K, such that $v\left(\frac{x_{r+1}}{H} - c\right) > 0$. Let

$$x_{r+1}^{(1)} = x_{r+1} - cH,$$

whence $v(x_{r+1}) < v(x_{r+1}^{(1)})$.

Let $f_1(x_1, \cdots, x_r, x_{r+1}^{(1)}) = f(x_1, \cdots, x_r, x_{r+1}^{(1)} + cH)$. We now deal with the polynomial f_1 in the same way as we dealt with f. We note that the multiplicity of the root $x_{r+1}^{(1)} = 0$ is still s, both in the case of the theorem and of the lemma. If our Cremona transformation, applied to $x_1, \cdots, x_r, x_{r+1}^{(1)}$, again fails to achieve a reduction of s, then we must have $v(x_{r+1}^{(1)}) = v(H_1)$, $H_1 \subset K[x_1, \cdots, x_r]$. We then determine a constant c_1 such that $v\left(\frac{x_{r+1}^{(1)}}{H_1} - c_1\right) > 0$, and we put $x_{r+1}^{(2)} = x_{r+1}^{(1)} - c_1H_1$, whence $v(x_{r+1}^{(1)}) < v(x_{r+1}^{(2)})$, and so we continue indefinitely.

Let us suppose that we get in this fashion a sequence of polynomials $f_i(x_1, \cdots, x_r, x_{r+1}^{(i)})$ for which no immediate reduction of s is possible by the method of section 4. Here

$$x_{r+1}^{(i)} = x_{r+1}^{(i-1)} - c_{i-1}H_{i-1}(x_1, \cdots, x_r), \tag{67}$$

$$f_i(x_1, \cdots, x_r, x_{r+1}^{(i)}) = f_{i-1}(x_1, \cdots, x_r, x_{r+1}^{(i)} + c_{i-1}H_{i-1}), \tag{67'}$$

$$v(x_{r+1}) < v(x_{r+1}^{(1)}) < \cdots < v(x_{r+1}^{(i)}) < \cdots, \tag{67''}$$

and where, in view of (67) and (67″), we have:

$$v(x_{r+1}^{(i)}) = v(H_i). \tag{68}$$

We assert *that such a sequence must be finite, provided $s \geqq 2$ or $s \geqq 1$, according as we deal with the theorem or with the lemma.* The proof of this assertion will complete the proof of both theorem and lemma.

In the case of the theorem we observe that $v\left(\frac{\partial f}{\partial x_{r+1}}\right) \geqq v(x_{r+1}^{s-1})$, since π_s must be a minimum value term (as a consequence of our hypothesis that our Cremona transformations fail to reduce the multiplicity s). Hence, if $s > 1$, then $v\left(\frac{\partial f}{\partial x_{r+1}}\right) > v(x_{r+1})$. Similarly, we find $v\left(\frac{\partial f_i}{\partial x_{r+1}^{(i)}}\right) > v(x_{r+1}^{(i)})$, and consequently, since $\frac{\partial f}{\partial x_{r+1}} = \frac{\partial f_i}{\partial x_{r+1}^{(i)}}$,

$$v\left(\frac{\partial f}{\partial x_{r+1}}\right) > v(x_{r+1}^{(i)}). \tag{69}$$

Now $v(x_{r+1})$, $v(x_{r+1}^{(1)})$, $\cdots$, $v(x_{r+1}^{(i)})$, $\cdots$, is a strictly ascending sequence of real numbers, and each element of the sequence is, by (68), the value of a polynomial in $x_1, \cdots, x_r$. It is easily seen (see [10], p. 659, footnote) that this last property implies that *if the sequence $v(x_{r+1}^{(i)})$ were infinite, then its limit would be $+\infty$.* But then (69) would imply that $\frac{\partial f}{\partial x_{r+1}}$ is necessarily the zero element of the field $\mathsf{K}(x_1, \cdots, x_r, x_{r+1})$ $(= \Sigma)$. This is impossible, since f is irreducible.

In the case of the lemma, we observe that $v(f) \geqq v(x_{r+1}^s)$. Hence, if $s \geqq 1$, then $v(f) \geqq v(x_{r+1})$. Similarly, $v(f_i) \geqq v(x_{r+1}^{(i)})$, and consequently

$$v(f) \geqq v(x_{r+1}^{(i)}).$$

Again, if the sequence $\{f_i\}$ was infinite, then by the same argument as above the above inequality would imply that $f = 0$, and this is impossible since $x_1, \cdots, x_r, x_{r+1}$ are algebraically independent.

III. The general case

6. Let $\alpha_1, \cdots, \alpha_{r+1}$ be the B-residues of $x_1, \cdots, x_{r+1}$ respectively, and let $\alpha_{i1} (= \alpha_i), \alpha_{i2}, \cdots, \alpha_{ig}$ be the conjugates of α_i over $\mathsf{K}(i = 1, 2, \cdots, r + 1)$. As before, we denote by R the field $\mathsf{K}(\alpha_{11}, \cdots, \alpha_{1g}, \cdots, \alpha_{r+1,1} \cdots, \alpha_{r+1,g})$. By a linear transformation of the coördinates $x_1, \cdots, x_{r+1}$, with coefficients in K, we may arrange matters so that, for each i, the g conjugate elements $\alpha_{i1}, \cdots, \alpha_{ig}$ be distinct. We write our polynomial f as a polynomial in $x_{r+1} - \alpha_{r+1}$:

$$f = a_0(x_1, \cdots, x_r) + a_1(x_1, \cdots, x_r)(x_{r+1} - \alpha_{r+1}) + \cdots$$
$$+ a_\nu(x_1, \cdots, x_r)(x_{r+1} - \alpha_{r+1})^\nu = \pi_0 + \pi_1 + \cdots + \pi_\nu .$$

By hypothesis, the coefficients of f are in R. Since $\alpha_{r+1} \subset \mathsf{R}$, it follows that the coefficients of the polynomials $a_0, a_1, \cdots, a_\nu$ are also in R, i.e. in the field generated by the B-residues of $x_1, \cdots, x_r$ and by their conjugates. By our induction, we therefore may apply the lemma to the product $a_0 a_1 \cdots a_\nu$. We have therefore a Cremona transformation from $x_1, \cdots, x_r$ to new variables $X_1, \cdots, X_r$, where the x's are polynomials in the X's, with coefficients in K, and such that each polynomial a_i assumes the form:

$$a_i(x_1, \cdots, x_r) = \Delta_1^{\mu_{1i}} \cdots \Delta_m^{\mu_{mi}} b_i(X_1, \cdots, X_r).$$

Here $\Delta_1, \cdots, \Delta_m$ are polynomials of the form (21) and $v(b_i) = 0$. Moreover, if $\gamma_1, \cdots, \gamma_r$ denote the B-residues of $X_1, \cdots, X_r$ and if we denote by Ω the field generated by these residues and by their conjugates (over K), then the polynomials b_i are in $\Omega[X_1, \cdots, X_r]$ (since this is true of $\Delta_1, \cdots, \Delta_m$).

The $\nu + 1$ terms π_i are, to within trivial factors b_i (i.e. factors having value zero in B), power products of $\Delta_1, \cdots, \Delta_m, x_{r+1} - \alpha_{r+1}$. The field Ω contains the residues $\alpha_1, \cdots, \alpha_r$ and their conjugates, since $x_1, \cdots, x_r$ are polynomials in $X_1, \cdots, X_r$, with coefficients in K. Hence this field Ω also contains the B-residue α_{r+1} of x_{r+1} and its g conjugates, since the g conjugates $\alpha_{i1}, \cdots, \alpha_{ig}$

are distinct. Now if we assume that the value of $x_{r+1} - \alpha_{r+1}$ is rationally independent on $\tau_1^*, \cdots, \tau_m^*(\tau_i^* = v(\Delta_i) = v(X_i - \gamma_i))$, then we can apply Theorem 3 and our lemma follows immediately by the same argument as that used in the preceding special case (C II 2). It is only necessary to recall from B III 12 that $\Delta_1, \cdots, \Delta_m, x_{r+1} - \alpha_{r+1}$ transform under $T^{(h)}$ as $X_1 - \gamma_1, \cdots, X_m - \gamma_m, x_{r+1} - \alpha_{r+1}$, provided h is sufficiently high.

7. If the value of $x_{r+1} - \alpha_{r+1}$ is rationally dependent on $\tau_1^*, \cdots, \tau_m^*$, our problem will be to achieve a reduction similar to the reduction achieved in the preceding special case (C II 4, 5). We go back to the original polynomial $f(x_1, \cdots, x_r, x_{r+1})$. We assume first of all, both in the case of the theorem and of the lemma, that $f(\alpha_1, \cdots, \alpha_r, x_{r+1})$ is not identically zero. In the case of the theorem we write f as a polynomial in $x_1 - \alpha_1, \cdots, x_r - \alpha_r, x_{r+1} - \alpha_{r+1}$. Let f begin with terms of lowest degree s in these differences. Then *without loss of generality we may assume that α_{r+1} is an s-fold root of $f(\alpha_1, \cdots, \alpha_r, x_{r+1})$.* All these assumptions can be satisfied by applying a preliminary linear transformation, with coefficients in K, to the elements $x_1, \cdots, x_{r+1}$.

In the case of the lemma we decompose f into its *irreducible factors* in $\mathsf{R}[x_1, \cdots, x_{r+1}]$ and for each irreducible factor φ_i we make the same assumptions as we have made above for f in the case of the theorem, namely: if φ_i begins with terms of lowest degree s_i in the differences $x_1 - \alpha_1, \cdots, x_{r+1} - \alpha_{r+1}$, then α_{r+1} is an s_i-fold root of $\varphi_i(\alpha_1, \cdots, \alpha_r, x_{r+1})$. To indicate the fact that φ_i begins with terms of lowest degree s_i, we shall say that *φ_i has an s_i-fold point at the center of the valuation B.* Similarly for f and s. In the case of the theorem, if $s = 1$, and in the case of the lemma, if all s are zero, there is nothing to prove.[19]

For convenience we indicate by φ a typical irreducible factor of f. We write f—or each φ (if we are dealing with the lemma)—as a polynomial in $x_{r+1} - \alpha_{r+1}$:

$$f(\text{or } \varphi) = a_0(x_1, \cdots, x_r) + a_1(x_1, \cdots, x_r)(x_{r+1} - \alpha_{r+1}) + \cdots + a_\nu(x_1, \cdots, x_r)(x_{r+1} - \alpha_{r+1})^\nu.$$

By our induction we may apply the lemma to the product $a_0 a_1 \cdots a_\nu$, or, if we are dealing with the lemma, to the product of *all* the coefficients a_i of *all* the irreducible factors φ_i. Then f—or each irreducible factor φ—assumes the form (we replace x_{r+1} by x_{m+1}):

$$f = \pi_0 + \pi_1 + \cdots + \pi_\nu,$$

where

$$\pi_i = b_i(X_1, \cdots, X_m, X_{m+2}, \cdots, X_{r+1}).\ \Delta_1^{\mu_{1i}} \cdots \Delta_m^{\mu_{mi}} \cdot (x_{r+1} - \alpha_{r+1})^i.$$

Here $\Delta_1, \cdots, \Delta_m$ are polynomials in $X_1, \cdots, X_m$ of type (21), and $v(b_i) = 0$.

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[19] In the case of the theorem, if $s = 1$, the derivative $\partial f/\partial x_{r+1}$ is $\neq 0$ at the center of B, and hence the center is a simple point ([11], p. 214, Theorem 11).

Moreover, the coefficients of b_i and of the Δ_i are in the field Ω of the B-residues $\gamma_1, \cdots, \gamma_m, \gamma_{m+2}, \cdots, \gamma_{r+1}$ of the X's. Let $v(\Delta_i) = \tau_i$, $v(x_{m+1} - \alpha_{m+1}) = \tau_{m+1}$. We have to assume that τ_{m+1} is rationally dependent on $\tau_1, \cdots, \tau_m$.

The polynomials $\Delta_1, \cdots, \Delta_m, x_{m+1} - \alpha_{m+1}$ form a normal set for the variables $X_1, \cdots, X_m, x_{m+1}$. We apply Theorem 4 (B III 15). We shall continue to denote the new variables of that theorem by letters X. We have:

$$\pi_i = B_i(X_1, \cdots, X_m, X_{m+1}, \cdots, X_{r+1})\Delta_1^{\mathbf{M}_{1i}} \cdots \Delta_m^{\mathbf{M}_{mi}}\Delta_{m+1}^{\mathbf{M}_{m+1,i}},$$

where the $\mathbf{M}_{ji}$ are given by the relations (32), with μ_{ji} instead of μ_j. The factors B now depend on all the $r + 1$ variables X_i, but in view of (58′) and of property (d) of Theorem 4, we have

$$B_i(\gamma_1, \cdots, \gamma_m, X_{m+1}, \gamma_{m+2}, \cdots, \gamma_{r+1}) = \text{const.} \neq 0. \tag{70}$$

Let $\pi_\alpha, \pi_\beta, \cdots \pi_\delta$ be the terms of lowest value. We know that Δ_{m+1} has value zero and that the values of $\Delta_1, \cdots, \Delta_m$ are rationally independent. Recalling what we said in section 12 concerning the behavior of the polynomials $\Delta_1, \cdots, \Delta_m$ under a Cremona transformation of type $T^{(h)}$, and taking into account Theorem 3, we conclude that by a Cremona transformation $T^{(h)}$ applied by $X_1, \cdots, X_m$ only, we can force the exponents $\mathbf{M}_{j\alpha}, \mathbf{M}_{j\beta}, \cdots, \mathbf{M}_{j\delta}$ to become equal, for all $j = 1, 2, \cdots, m$, and to be less than $\mathbf{M}_{ji}$ for any i such that π_i is not a lowest value term. We can do that for all irreducible factors φ_i of f simultaneously, in the case of the lemma. This transformation does not affect the exponents $\mathbf{M}_{m+1,i}$ and the validity of (70). Suppose that this has already been done. Then, if we put $\mathbf{M}_{j\alpha} = \mathbf{M}_{j\beta} = \cdots \mathbf{M}_{j\delta} = \mathbf{M}_j$, and if we recall the relation (35), we find that f(or each φ) has the following form:

$$f(\text{or } \varphi) = \Delta_1^{\mathbf{M}_1} \cdots \Delta_m^{\mathbf{M}_m} F(X_1, \cdots, X_{r+1}), \tag{71}$$

where

$$F = \Delta_{m+1}^{\mathbf{M}_{m+1}}\psi(X_1, \cdots, X_{r+1}) + \Delta_1\Delta_2 \cdots \Delta_m G(X_1, \cdots, X_{r+1}).$$

Here

$$\psi = B_\alpha + B_\beta\Delta_{m+1}^{(\beta-\alpha)/d} + \cdots + B_\delta\Delta_{m+1}^{(\delta-\alpha)/d} \quad \text{and} \quad \mathbf{M}_{m+1} = \mathbf{M}_{m+1,\alpha}, \tag{71′}$$

if $d > 0$, and

$$\psi = B_\alpha\Delta_{m+1}^{(\alpha-\delta)/d} + B_\beta\Delta_{m+1}^{(\beta-\delta)/d} + \cdots + B_\delta \text{ and } \mathbf{M}_{m+1} = \mathbf{M}_{m+1,\delta}, \tag{71″}$$

if $d < 0$. We continue to denote by $\gamma_1, \cdots, \gamma_{r+1}$ the B-residues of $X_1, \cdots, X_{r+1}$, and by Ω the field generated by these residues and by their conjugates. The coefficients of the Δ_i and of the polynomial F are in Ω. Let γ_{r+1} be a root of multiplicity s' for the polynomial $F(\gamma_1, \cdots, \gamma_m, X_{m+1}, \gamma_{m+2}, \cdots, \gamma_{r+1})$. *Let us assume that* $s' < s$; we assume that this is so *for each irreducible factor* φ, if we are dealing with the lemma.

We consider first the situation in the case of the theorem. Our Cremona transformation from the x's to the X's has its coefficients in K. Let $F_1(X_1, \cdots, X_{r+1}) = 0$ be the new (irreducible) equation of the hypersurface (over K). The polynomial F has its coefficients in K and must be a factor of $\Delta_1^{M_1}\Delta_2^{M_2} \cdots \Delta_m^{M_m}F$. The polynomials Δ_i depend only on $X_1, \cdots, X_m$, and these m elements of Σ are *algebraically independent*, since the values $v(X_1 - \gamma_1), \cdots, v(X_m - \gamma_m)$ are rationally independent. On the other hand $F_1(X_1, \cdots, X_{r+1}) = 0$ in Σ. Consequently, for any $i = 1, 2, \cdots, m$, F_1 and Δ_i can have no common factors in $\Omega[X_1, \cdots, X_{r+1}]$ (as a matter of fact, F_1 and Δ_i remain relatively prime even if we pass from Ω to the algebraically closed field determined by Ω). *It follows that F_1 is a factor of F.* Since F does not vanish identically in X_{m+1} when we put $X_i = \gamma_i (i = 1, 2, \cdots, m, m + 2, \cdots, r + 1)$, the same holds true also for F_1. On the other hand, since γ_{m+1} is a root of multiplicity s' of $F(\gamma_1, \cdots, \gamma_m, X_{m+1}, \gamma_{m+2}, \cdots, \gamma_{r+1})$, F has at most an s'-fold point at the center of B (in the space of the X's). Hence also F_1 has at most an s'-fold point at the center of B, and since $s' < s$, we have achieved a reduction. Ultimately, we will get an hypersurface for which $s = 1$, and our main theorem will be proved in the case under consideration.[20]

Let us now consider the case of the lemma. We observe that $\Delta_1, \cdots, \Delta_m$ have each a simple point at the center of B; this follows from the expression (21) of the Δ_i. Hence our Cremona transformation has the effect of replacing each irreducible factor φ_i of f, by a factor F_i which has at the center of B (in the space of $X_1, \cdots, X_{r+1}$) a multiplicity s_i' which is less than the multiplicity s_i of φ_i at the center of B (in the space of $x_1, \cdots, x_{r+1}$). In addition, new irreducible factors Δ_i appear, but these have each *a simple point* at the center of $B(s = 1)$. If we now apply the same procedure to f, expressed in terms of the new variables $X_1, \cdots, X_{r+1}$, and if the reduction $(s' < s)$ succeeds at each step, then it is clear that ultimately f will assume the form (71) and in this F *will have value zero,* which establishes the lemma.

8. There remains the case in which $s' = s$ for f, or—in the case of the lemma—for at least one irreducible factor φ of f. The reasoning given in the special case (C II 4) can be repeated here. The equality $s' = s$ implies $\alpha = 0$,

[20] We add a remark which will be useful to us in the sequel. We know that in the ring $\mathsf{K}[X_1, \cdots, X_{r+1}]$ the polynomial $f(x_1, \cdots, x_{r+1})$ (expressed in terms of the X's) is divisible by $F_1(X_1, \cdots, X_{r+1})$. *We assert that f is not divisible by F_1^2* (the x's as well as the X's are now regarded temporarily as indeterminates). To see this, we observe that the prime principal ideal (F_1) in the ring $\mathsf{K}[X_1, \cdots, X_{r+1}]$ defines in the field $\mathsf{K}(X_1, \cdots, X_{r+1})$ a valuation of rank 1 and of dimension r. In this valuation the value of F_1 is 1. The prime ideal of this valuation in the smaller ring $\mathsf{K}[x_1, \cdots, x_{r+1}]$ is (f) (since $f \equiv 0(F_1)$ in $\mathsf{K}[X_1, \cdots, X_{r+1}]$). Hence the same valuation is defined by the principal ideal (f) in $\mathsf{K}[x_1, \cdots, x_{r+1}]$. Therefore also the value of f is 1, and this proves our assertion. Thus we may write the following identity (the x's and the X's being indeterminates related by our Cremona transformation): $f(x_1, \cdots, x_{r+1}) = F_1(X_1, \cdots, X_{r+1}) \cdot H(X_1, \cdots, X_{r+1})$, where $H \not\equiv 0(F_1)$.

$\beta = 1, \cdots, \delta = s$, and we conclude, as before, that $v(x_{r+1} - \alpha_{r+1}) = v(a_{s-1})$.[21] But at this stage there is a new complication. We can of course find a constant c in K^* such that

$$(72) \qquad v\left(\frac{x_{r+1} - \alpha_{r+1}}{a_{s-1}} - c\right) > 0.$$

The transformation used in C II 5 was the following: $x_{r+1} - ca_{s-1} = x_{r+1}^{(1)}$, from which it would now follow that $v(x_{r+1}^{(1)} - \alpha_{r+1}) > v(x_{r+1} - \alpha_{r+1})$. *But now this transformation cannot be used, since its coefficients are not in* K. To overcome this difficulty, we proceed as follows,

We first observe that the coefficients of a_{s-1} are in the field R. *We assert that also the constant c is in* R. This is a consequence of our hypothesis $s' = s$, and is shown as follows.

We divide the polynomial f (or φ) by $a_{s-1}^s a_s$. Since $v(a_{s-1}) = v(x_{r+1} - \alpha_{r+1})$, we have: $v(a_{s-1}^s a_s) = v(a_s \cdot (x_{r+1} - \alpha_{r+1})^s) = v(\pi_s)$, whence $v(a_{s-1}^s a_s) = v(\pi_s) = v(\pi_{s-1}) = \cdots = v(\pi_0)$. Hence the quotients $\pi_0/a_{s-1}^s a_s, \cdots, \pi_s/a_{s-1}^s a_s$ have value zero, while the quotients $\pi_i/a_{s-1}^s a_s$, $i > s$, have positive values. On the other hand, the B-residue of $f/a_{s-1}^s a_s$, in the case of the theorem, and the B-residue of $\varphi/a_{s-1}^s a_s$, in the case of the lemma, in zero.[22] Therefore, if we replace in $f/a_{s-1}^s a_s$ (or in $\varphi/a_{s-1}^s a_s$) each term $\pi_i/a_{s-1}^s a_s$ by its B-residue, and if we observe that $B(\pi_i/a_{s-1}^s a_s) = B\left(\frac{a_i}{a_{s-1}^{s-i} a_s} \cdot \left(\frac{x_{r+1} - \alpha_{r+1}}{a_{s-1}}\right)^i\right) = c^i \cdot B\left(\frac{a_i}{a_{s-1}^{s-i} a_s}\right)$, where c is the B-residue of $\frac{x_{r+1} - \alpha_{r+1}}{a_{s-1}}$ according to (72), we conclude that c satisfies the equation:

$$(73) \qquad d_0 + d_1 c + \cdots + d_{s-1} c^{s-1} + c^s = 0,$$

where

$$(74) \qquad d_i = B(a_i/a_{s-1}^{s-i} a_s), \qquad i = 0, 1, \cdots, s-1.$$

We have

$$\frac{\pi_i}{\pi_s} = \frac{a_i}{(x_{r+1} - \alpha_{r+1})^{s-i} a_s} = \frac{a_i}{a_{s-1}^{s-i} a_s} \cdot \left(\frac{a_{s-1}}{x_{r+1} - \alpha_{r+1}}\right)^{s-i},$$

whence, if we put

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[21] We return to the original notation: x_{r+1}, instead of x_{m+1}. The polynomials $a_1, \cdots, a_\nu$ are the coefficients of f or—in the case of the lemma—of that irreducible factor φ for which there is no reduction of s.

[22] This is obvious in the case of the theorem, since in that case $f = 0$. In the case of the lemma, our assertion follows from the hypothesis that there was no reduction of the multiplicity s. This hypothesis implies at any rate that the polynomial F in (71) has positive value in B, i.e. $v(F) > 0$. Since $v(\Delta_1^{M_1} \cdots \Delta_m^{M_m}) = v(\pi_\alpha) = v(\pi_\beta) = \cdots = v(\pi_\delta)$, it follows that $v(\varphi/\pi_\alpha) > 0$. In the present case we have $\pi_\alpha = \pi_0$ and $v(\pi_0) = v(a_{s-1}^s a_s)$. Hence $v(\varphi/a_{s-1}^s a_s) > 0$, i.e. $B(\varphi/a_{s-1}^s a_s) = 0$.

(75) $$d'_i = B(\pi_i/\pi_s), \qquad i = 0, 1, \cdots, s,$$

then

(75′) $$d_i = d'_i \cdot c^{s-i}, \qquad i = 0, 1, \cdots, s; d_s = 1.$$

On the other hand, from our hypothesis that the B-residue γ_{r+1} of X_{r+1} is an s-fold root of the polynomial $\psi(\gamma_1, \cdots, \gamma_r, X_{r+1})$ (see (71′) and (71″)), follows that if t is an indeterminate, then the polynomial $d'_0 + d'_1 t + \cdots + d'_{s-1}t^{s-1} + t^s$ is a perfect s^{th} power. Namely,

$$d'_0 + d'_1 t + \cdots + d'_{s-1}t^{s-1} + t^s = \left(t - \frac{1}{\gamma_{r+1}}\right)^s.$$

Hence, by (75′), also the polynomial $d_0 + d_1 t + \cdots + d_{s-1}t^{s-1} + t^s$ is a perfect s^{th} power, i.e., by (73),

$$d_0 + d_1 t + \cdots + d_{s-1}t^{s-1} + t^s = (t - c)^s.$$

Consequently,

$$c = -\frac{d_{s-1}}{s}, \quad \text{i.e., by (74),} \quad \text{for } i = s - 1,$$

$$c = -\frac{1}{s}\cdot B\left(\frac{1}{a_s}\right),$$

and since $v(a_s) = 0$, whence $a_s(\alpha_1, \cdots, \alpha_r) \neq 0$, we conclude that $c = -\dfrac{1}{s\cdot a_s(\alpha_1, \cdots, \alpha_r)}$, and therefore $c \subset \mathsf{R}$. This proves our assertion.

Now that we have recognized that the coefficients of the polynomial $c\cdot a_{s-1}(x_1, \cdots, x_r)$ are in the field R, we may apply Lemma 1 (B III 10), where we put $A_j = 1$ and where we take for $B_1, \cdots, B_g$ the polynomial $c\cdot a_{s-1}$ and its conjugates over K. There exists then a polynomial $H(x_1, \cdots, x_r)$ in $\mathsf{K}[x_1, \cdots, x_r]$, such that $H - c\cdot a_{s-1}$, written as a polynomial in $x_1 - \alpha_1$, $x_r - \alpha_r$, begins with terms of degree $\geqq \rho_0$. If we now take ρ_0 sufficiently high, then we will have: $v(H - ca_{s-1}) > v(x_{r+1} - \alpha_{r+1} - ca_{s-1})$. If we put

(76) $$x_{r+1} = x^{(1)}_{r+1} + H(x_1, \cdots, x_r),$$

then $v(x^{(1)}_{r+1} - \alpha_{r+1}) = v(x_{r+1} - H - \alpha_{r+1}) \geqq \min.\{v(x_{r+1} - \alpha_{r+1} - ca_{s-1}), v(H - c\cdot a_{s-1})\} = v(x_{r+1} - \alpha_{r+1} - ca_{s-1}) > v(x_{r+1} - \alpha_{r+1})$, i.e.

(76′) $$v(x^{(1)}_{r+1} + \alpha_{r+1}) > v(x_{r+1} - \alpha_{r+1}) = v(H).$$

Thus we have the transformation (76) *whose coefficients are in* K and, in view of (76′) this transformation plays the same role as the transformation (67) in the special case. Moreover, by (76′), the B-residue of $x^{(1)}_{r+1}$ is the same as that of x_{r+1}, whence the field R remains the same. Finally, the integer s is obviously unaltered by the transformation (76). The rest of the proof is now the same in C II 5.

IV. Valuation of arbitrary rank and dimension

9. We assume that the uniformization theorem U_3 has already been proved for zero-dimensional valuations of rank σ. We proceed to prove the theorem, first for valuations of rank σ and of any dimension, and then for zero-dimensional valuations of rank $\sigma + 1$. This will complete the proof of the general uniformization theorem U_3.

Let B be an s-dimensional valuation of rank σ, and let V be a given projective model of Σ. Let $(\xi_1, \cdots, \xi_n)$ be the general point of V and let $\mathfrak{o} = K[\xi_1, \cdots, \xi_n]$. We may assume that $\mathfrak{o}$ belongs to the valuation ring $\mathfrak{B}$ of B. The center of B on V is a subvariety W of dimension $s' \leqq s$, given by a prime s'-dimensional ideal $\mathfrak{p}$ in $\mathfrak{o}$. Of the elements $\xi_1, \cdots, \xi_n$, s' and only s', say $\xi_1, \cdots, \xi_{s'}$, are algebraically independent mod $\mathfrak{p}$, and their B-residues are algebraically independent elements of the residue field of B. Suppose that $s' < s$. We can choose in Σ $s - s'$ elements $\eta_1, \cdots, \eta_{s-s'}$, such that the B-residues of $\xi_1, \cdots, \xi_{s'}, \eta_1, \cdots, \eta_{s-s'}$ be algebraically independent (over K). Let V' be the projective model of Σ whose general point is $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{s-s'})$, and let $\mathfrak{o}' = K[\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{s-s'}]$. The ring $\mathfrak{o}'$ is contained in the valuation ring of B. The center W' of B on V' is exactly of dimension s, since $\mathfrak{o}'$ contains s elements whose B-residues are algebraically independent. The quotient ring $Q(W')$ contains $Q(W)$, since $\mathfrak{o}$ is a subring of $\mathfrak{o}'$. Hence it is sufficient to prove the uniformization theorem for B and V'. We may therefore assume that the center W of B on the original variety V is of dimension s.

Let the s ξ's whose B-residues are algebraically independent over K be $\xi_1, \cdots, \xi_s$. We take as new ground field the field $\Delta = K(\xi_1, \cdots, \xi_s)$. Over this new ground field the general point $(\xi_1, \cdots, \xi_n)$ defines an $(r - s)$-dimensional variety V^*. The valuation B is now zero-dimensional, of rank σ. Its origin on V^* is a point P^*, and we have: $Q(P^*) = Q(W)$ ([11], p. 219). By our induction, there exists a projective model V_1^* of the field Σ/Δ on which the origin of B is a simple point P_1^*, such that $Q(P_1^*) \supseteqq Q(P^*)$. Let $(\eta_1, \cdots, \eta_m)$ be the general point of V_1^*. We may assume that the η's belong to the valuation ring of B. Let V_1 be the projective model of Σ/K whose general point is $(\xi_1, \cdots, \xi_s, \eta_1, \cdots, \eta_m)$, and let W_1 be the center of B on V_1. We have: $Q(W_1) = Q(P_1^*)$, and W_1 is a simple subvariety of V_1 ([11], p. 219–220). Since $Q(W_1) \supseteqq Q(P^*) = Q(W)$, the proof in the present case is complete.

10. Let now B be a zero-dimensional valuation of rank $\sigma + 1$, and let V be the given projective model of Σ/K. The valuation B is composite with a valuation B_1, of rank σ and of a certain dimension s. Let P and W be the centers on V of B and B_1 respectively. Let $\xi_1, \cdots, \xi_n$ be the general point of V and let $\mathfrak{o} = K[\xi_1, \cdots, \xi_n]$, where we assume that the ξ's belong to the valuation ring of B.

Let, by the preceding proof, V' be a projective model of Σ/K on which the center of B_1 is a simple subvariety W' such that $Q(W') \supseteqq Q(W)$, and let

$\eta_1, \cdots, \eta_m$ be the general point of V'. Again we may assume that the η's belong to the valuation ring of B (note that the quotient ring $Q(W')$ is invariant under projective transformations of the coördinates η_i). *A fortiori*, the η_i will belong to the valuation ring of B_1. We consider the projective model $\bar{V}$ of Σ/K whose general point is $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m)$. The ring $\bar{\mathfrak{o}} = \mathsf{K}[\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m]$ is the join of the two rings $\mathfrak{o} = \mathsf{K}[\xi_1, \cdots, \xi_n]$, $\mathfrak{o}' = \mathsf{K}[\eta_1, \cdots, \eta_m]$. Since $\mathfrak{o} \subseteq Q(W) \subseteq Q(W')$ and since $\mathfrak{o}' \subseteq Q(W')$, it follows that $\bar{\mathfrak{o}} \subseteq Q(W')$. On the other hand $\mathfrak{o}' \subseteq \bar{\mathfrak{o}}$. From this it follows that if $\bar{W}$ is the center of B_1 on $\bar{V}$, then $Q(\bar{W}) = Q(W')$, (and hence $\bar{W}$ is a simple subvariety of $\bar{V}$). Since $\mathfrak{o} \subseteq \bar{\mathfrak{o}}$, and since the center $\bar{P}$ of B on $\bar{W}$ is a point at finite distance, it follows that $Q(P) \subseteq Q(\bar{P})$. *It is therefore permissible to replace in the proof the variety V by the variety $\bar{V}$. We may therefore suppose that the center W of B_1, on the original variety V, is simple.*

11. Let ρ be the dimension of W. We choose in $\mathfrak{o}(= \mathsf{K}[\xi_1, \cdots, \xi_n])$ a set of $r + 1$ elements $\eta_1, \cdots, \eta_{r+1}$, such that: (a) $\mathfrak{o}$ is integrally dependent on $\mathsf{K}[\eta_1, \cdots, \eta_r]$; (b) η_{r+1} is a primitive element of Σ with respect to $\mathsf{K}(\eta_1, \cdots, \eta_r)$; (c) $\eta_{\rho+1}, \cdots, \eta_r$ are uniformizing parameters of the simple subvariety W; (d) if $\mathfrak{p}$ is the prime ideal of W in $\mathfrak{o}$ and if $F(\eta_1, \cdots, \eta_{r+1}) = 0$ is the irreducible relation between $\eta_1, \cdots, \eta_{r+1}$, then $F'_{\eta_{r+1}} \not\equiv 0(\mathfrak{p})$ ([11], p. 214). Let V' be the hypersurface $F(\eta_1, \cdots, \eta_{r+1}) = 0$, and let W' and P' be the centers of B_1, B respectively, on V'. Exactly as in A IV it is shown that it is sufficient to prove the uniformization theorem for V' and B.

The prime ideal $\mathfrak{p}'$ of W' in $\mathsf{K}[\eta_1, \cdots, \eta_{r+1}]$ is such that $\mathfrak{p} \cap \mathsf{K}[\eta_1, \cdots, \eta_{r+1}] = \mathfrak{p}'$. From (d) it follows that $F'_{\eta_{r+1}} \not\equiv 0(\mathfrak{p}')$, whence W' is a simple subvariety of V'. *Hence we may assume that our original V is an hypersurface*

$$f(\xi_1, \cdots, \xi_r, \xi_{r+1}) = 0,$$

that $\xi_{\rho+1}, \cdots, \xi_r$ are uniformizing parameters of W, and that $f'_{\xi_{r+1}} \not\equiv 0(\mathfrak{p})$, where $\mathfrak{p}$ is the prime (ρ-dimensional) ideal of W in $\mathsf{K}[\xi_1, \cdots, \xi_{r+1}]$. Moreover, ξ_{r+1} is integrally dependent on $\xi_1, \cdots, \xi_r$.

12. The ideal $\mathfrak{p}$ is ρ-dimensional. Since $\xi_i \equiv 0(\mathfrak{p})$, $i = \rho + 1, \cdots, r$, and since ξ_{r+1} is integrally dependent on $\mathsf{K}[\xi_1, \cdots, \xi_r]$, it follows that $\xi_1, \cdots, \xi_\rho$ are algebraically independent mod $\mathfrak{p}$. Let $g(\xi_1, \cdots, \xi_\rho, \xi_{r+1}) \equiv 0(\mathfrak{p})$ be the irreducible congruence which $\xi_1, \cdots, \xi_\rho, \xi_{r+1}$ satisfy mod $\mathfrak{p}$. Since $f(\xi_1, \cdots, \xi_{r+1}) \equiv f(\xi_1, \cdots, \xi_\rho, 0, \cdots, 0, \xi_{r+1})$ (mod $\mathfrak{p}$), the polynomial $f(\xi_1, \cdots, \xi_\rho, 0, \cdots, 0, \xi_{r+1})$ is divisible by $g(\xi_1, \cdots, \xi_\rho, \xi_{r+1})$. It is *not* divisible by g^2, since $f'_{\xi_{r+1}} \not\equiv 0(\mathfrak{p})$. Let therefore

$$\begin{aligned} &f(\xi_1, \cdots, \xi_\rho, 0, \cdots, 0, \xi_{r+1}) \\ &\qquad = g(\xi_1, \cdots, \xi_\rho, \xi_{r+1}) \cdot \varphi(\xi_1, \cdots, \xi_\rho, \xi_{r+1}), \qquad \varphi \not\equiv 0(g). \end{aligned} \tag{77}$$

If then $x_1, \cdots, x_{r+1}$ are indeterminates, then we have identically:

$$f(x_1, \cdots, x_{r+1}) = g(x_1, \cdots, x_\rho, x_{r+1}) \cdot \varphi(x_1, \cdots, x_\rho, x_{r+1}) \tag{78}$$
$$+ a_1 x_{\rho+1} + \cdots + a_{r-\rho} x_r,$$

where $a_i \subset \mathsf{K}[x_1, \cdots, x_{r+1}]$.

The valuation B_1, with which B is composite, is of rank σ. Hence B defines *in the residue field of* B_1 a zero-dimensional valuation of *rank* 1. Since $\mathfrak{p}$ is the prime ideal of B_1 in $\mathsf{K}[\xi_1, \cdots, \xi_{r+1}]$, the B-residues of $\xi_1, \cdots, \xi_\rho, \xi_{r+1}$ are the residues of these elements mod $\mathfrak{p}$. Let $\bar{\xi}_1, \cdots, \bar{\xi}_\rho, \bar{\xi}_{r+1}$ be these residues. We consider the ring $\bar{\mathfrak{o}} = \mathsf{K}[\bar{\xi}_1, \cdots, \bar{\xi}_\rho, \bar{\xi}_{r+1}]$ and its quotient field $\bar{\Sigma}$. This field is of degree of transcendency ρ over K, and the hypersurface

$$g(x_1, \cdots, x_\rho, x_{r+1}) = 0. \tag{79}$$

is a projective model of $\bar{\Sigma}$. The general point of this hypersurface is $(\bar{\xi}_1, \cdots, \bar{\xi}_\rho, \bar{\xi}_{r+1})$. The field $\bar{\Sigma}$ is contained in the residue field of B_1. Hence the original zero-dimensional valuation B of Σ induces a zero-dimensional, *rank* 1, valuation $\bar{B}$ of $\bar{\Sigma}$. By our main theorem, there exists a Cremona transformation

$$x_i = P_i(y_1, \cdots, y_\rho, y_{r+1}), \qquad i = 1, 2, \cdots, \rho, r+1, \tag{80}$$

where $P_i \subset \mathsf{K}[y_1, \cdots, y_\rho, y_{r+1}]$, by which the hypersurface (79) is transformed into an hypersurface

$$G(y_1, \cdots, y_\rho, y_{r+1}) = 0, \tag{81}$$

on which the center of $\bar{B}$ is at a simple point. Moreover, the coördinates $\bar{\eta}_1, \cdots, \bar{\eta}_\rho, \bar{\eta}_{r+1}$ of the general point of (81) are in the valuation ring of $\bar{B}$. The $\bar{B}$-residues $\alpha_1, \cdots, \alpha_\rho, \alpha_{r+1}$ of $\bar{\eta}_1, \cdots, \bar{\eta}_\rho, \bar{\eta}_{r+1}$ are the coördinates of the center of B on the hypersurface (81). Since the center is a simple point, we may assume that

$$G'_{y_{r+1}}(\alpha_1, \cdots, \alpha_\rho, \alpha_{r+1}) \neq 0. \tag{82}$$

The polynomial $g(x_1, \cdots, x_\rho, x_{r+1})$, expressed in terms of $y_1, \cdots, y_\rho, y_{r+1}$, assumes the form

$$g = G(y_1, \cdots, y_\rho, y_{r+1}) \cdot H(y_1, \cdots, y_\rho, y_{r+1}),$$

where (see footnote[20]),

$$H \not\equiv 0(G). \tag{83}$$

If we now put $\varphi(x_1, \cdots, x_\rho, x_{r+1}) = \psi(y_1, \cdots, y_\rho, y_{r+1})$ and $a_i(x_1, \cdots, x_{r+1}) = A_i(y_1, \cdots, y_\rho, x_{\rho+1}, \cdots, x_r, y_{r+1})$, the identity (78) assumes the following form:

$$f(x_1, \cdots, x_{r+1}) = GH\psi + A_1 x_{\rho+1} + \cdots + A_{r-\rho} x_r. \tag{84}$$

If we replace in (80) the indeterminates $x_1, \cdots, x_\rho, x_{r+1}$ by the elements $\xi_1, \cdots, \xi_\rho, \xi_{r+1}$ of the field Σ, we have to replace the indeterminates $y_1, \cdots, y_\rho, y_{r+1}$ by well-defined elements $\eta_1, \cdots, \eta_\rho, \eta_{r+1}$ of the field Σ, such that:

$\xi_i = P_i(\eta_1, \cdots, \eta_\rho, \eta_{r+1})$. The B_1-residues of the η's are the coördinates $\bar{\eta}_1, \cdots, \bar{\eta}_\rho, \bar{\eta}_{r+1}$ of the general point of the hypersurface (81). In view of (83), it follows that the B_1-residue of $H(\eta_1, \cdots, \eta_\rho, \eta_{r+1})$ is $\neq 0$, whence $v_{B_1}(H(\eta_1, \cdots, \eta_\rho, \eta_{r+1})) = 0$. Similarly, since $\varphi \not\equiv 0(g)$ (see (77)), also $v_{B_1}(\varphi(\xi_1, \cdots, \xi_\rho, \xi_{r+1})) = v_{B_1}(\psi(\eta_1, \cdots, \eta_\rho, \eta_{r+1}) = 0$. Hence $v_{B_1}(H\psi) = 0$. Since $\xi_{\rho+1}, \cdots, \xi_r$ are the uniformizing parameters of the center W of B_1 on the hypersurface $f = 0$, their values in B_1 are positive. Since B is composite with B_1, it follows that if we put

$$\eta_i = \xi_i/H\psi, \qquad i = \rho + 1, \cdots, r, \tag{85}$$

then $\eta_{\rho+1}, \cdots, \eta_r$ *have positive values in* B. We have now $r + 1$ elements of $\Sigma: \eta_1, \cdots, \eta_{r+1}$. The elements $\xi_1, \cdots, \xi_{r+1}$ are polynomials in the η's, with coefficients in K. We have namely: $\xi_i = P_i(\eta_1, \cdots, \eta_\rho, \eta_{r+1})$, $i = 1, 2, \cdots, \rho, r + 1$, and $\xi_i = H(\eta_1, \cdots, \eta_\rho, \eta_{r+1}) \cdot \psi(\eta_1, \cdots, \eta_\rho, \eta_{r+1}) \cdot \eta_i$, $i = \rho + 1, \cdots, r$. The hypersurface $f(\xi_1, \cdots, \xi_{r+1}) = 0$ is now transformed into an hypersurface $F(\eta_1, \cdots, \eta_{r+1}) = 0$, where, in view of (84) and (85):

$$F = G(\eta_1, \cdots, \eta_\rho, \eta_{r+1}) + B_1\eta_{\rho+1} + \cdots + B_{r-\rho}\eta_r. \tag{86}$$

The B-residues of the η's are: $B(\eta_i) = \bar{B}(\bar{\eta}_i) = \alpha_i$, $i = 1, 2, \cdots, \rho, r + 1$; $B(\eta_i) = 0$, $i = \rho + 1, \cdots, r$. The point P_1^* $(\alpha_1, \cdots, \alpha_\rho, 0, \cdots, 0, \alpha_{r+1})$ is the center P^* of B on the hypersurface $F = 0$ (or, more precisely: the center P^* consists of that point P_1^* and of its conjugates over K). From the form (86) of F it follows immediately, in view of (82), that P^* is a simple point.

If P is the center of B on the original hypersurface $f = 0$, the quotient ring of P is a subring of the quotient ring of P^*, since $\xi_1, \cdots, \xi_{r+1}$ are polynomials in $\eta_1, \cdots, \eta_{r+1}$. This completes the proof of the uniformization theorem.

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